Economics 2301

Lecture 2

Limits
Limits and Continuity

- It is often necessary to evaluate a function as its argument approaches some value.
- The **limit** of a function as its argument approaches some number \( a \) is simply the number the function’s value approaches as the argument approaches \( a \), either from smaller values of \( a \), giving the **left-hand limit**, or from larger values of \( a \), giving the **right-hand limit**.
The left-hand limit of a function $f(x)$ as its argument approaches some number $a$, written as

$$\lim_{x \to a^-} f(x)$$

exists and is equal to $L^L$ if, for any arbitrarily small number $\epsilon$, there exists a small number $\delta$ such that

$$|f(x) - L^L| < \epsilon$$

whenever $a - \delta < x < a$. 

Left-hand Limit
The right-hand limit of a function $f(x)$ as its argument approaches some number $a$, written as

$$\lim_{x \to a^+} f(x)$$

exists and is equal to $L^R$ if, for any arbitrarily small number $\varepsilon$, there exists a small number $\delta$ such that

$$| f(x) - L^R | < \varepsilon$$

whenever $a < x < a + \delta$. 
When the left-hand limit equals the right-hand limit, we can simplify the notation by suppressing the superscripts and defining

\[
\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x)
\]
Infinite Limits

The limit of a function as its argument approaches $a$ equals positive infinity if the value of the function increases without bound, and equals negative infinity if the value decreases without bound.

$$
\lim_{x \to a} f(x) = +\infty \text{ if, for every } N > 0, \text{ there exits a } \delta > 0 \\
\text{so that } f(x) > N \text{ whenever } a - \delta < x < a + \delta.
$$

Also, we have $\lim_{x \to a} f(x) = -\infty$ if, for every $N < 0$, there is a $\delta > 0$ so that $f(x) < N$ whenever $a - \delta < x < a + \delta$. 
Left-hand limit ≠ Right-hand limit
Rules for Evaluating Limits

**Rule 1**

\[
\lim_{x \to 0^-} m(k + x) = \lim_{x \to 0^+} m(k + x) = mk
\]

**Rule 2**

\[
\lim_{x \to \infty} \frac{k}{m \cdot x + h} = 0
\]
Example 1 of Limit

\[
\lim_{x \to 0^+} \left( \frac{x^2 - 9}{x + 3} \right) = \lim_{x \to 0^+} (x - 3) = -3
\]
Example 2 of Limit

$$\lim_{{x \to \infty}} \left( \frac{10}{5x} + 20 \right) = 20$$
Our previous function

\[ \lim_{x \to 0^+} \left( \frac{2}{x^3} \right) = \infty \]

\[ \lim_{x \to 0^-} \left( \frac{2}{x^3} \right) = -\infty \]
A continuous univariate function has no "breaks" or "jumps."

A function \( f(x) \) is continuous at \( x = a \), where is in the domain of \( f \), if the left- and right-hand limits at \( x = a \) exist and are equal.

\[
\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)
\]

and the limit as \( x \to a \) equals the value of the function at that point,

\[
\lim_{x \to a} f(x) = f(a)
\]
Figure 2.5 Functions That Are Not Continuous

(a) $f(x)$

(b) Total cost

(c) $f(x)$
Our function that was not continuous

\[ y = \frac{2}{x^3} \]
Our noncontinuous function

We know from a previous slide that

\[
\lim_{x \to 0^+} \left( \frac{2}{x^3} \right) = \infty \quad \text{and} \quad \lim_{x \to 0^-} \left( \frac{2}{x^3} \right) = -\infty
\]

\[
f(x) = \frac{2}{x^3} \quad \text{at} \quad x = 0 \quad \text{is} \quad \frac{2}{0} \quad \text{which is not defined.}
\]
A vertical asymptote of a function occurs at a point when either a left-hand limit or a right-hand limit approaches positive infinity or negative infinity at that point. A function is discontinuous at a point where there is a vertical asymptote.