Economics 2301

Lecture 5

Introduction to Functions Cont.

Necessary & Sufficient Conditions

- Symbol $\Rightarrow$ means “implies”
- P$\Rightarrow$Q means “if P then Q,” “P implies Q,” “P only if Q.” P is a **sufficient condition** for Q, for if P holds then if follows that Q holds.
- This expression also shows that Q is a **necessary condition** for P cause P can only hold if Q holds.

Example of Necessary and Sufficient Conditions

- f strictly concave $\Rightarrow$ f concave

If and only if

$$f \text{ is concave } \Leftrightarrow f\left(\lambda x_A + (1-\lambda) x_B\right) \geq \lambda f\left(x_A\right) + (1-\lambda) f\left(x_B\right)$$

Power Functions

A power function takes the general form

$$f(x)=kx^P$$

k and p are constants.

The parameter p is the exponent of the funciton.

Rules of Exponents

$$x^0 = 1$$

$$x^1 = x$$

$$x^{-p} = \frac{1}{x^p}$$

$$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

$$x^a x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$\left(x^a\right)^b = x^{ab}$$

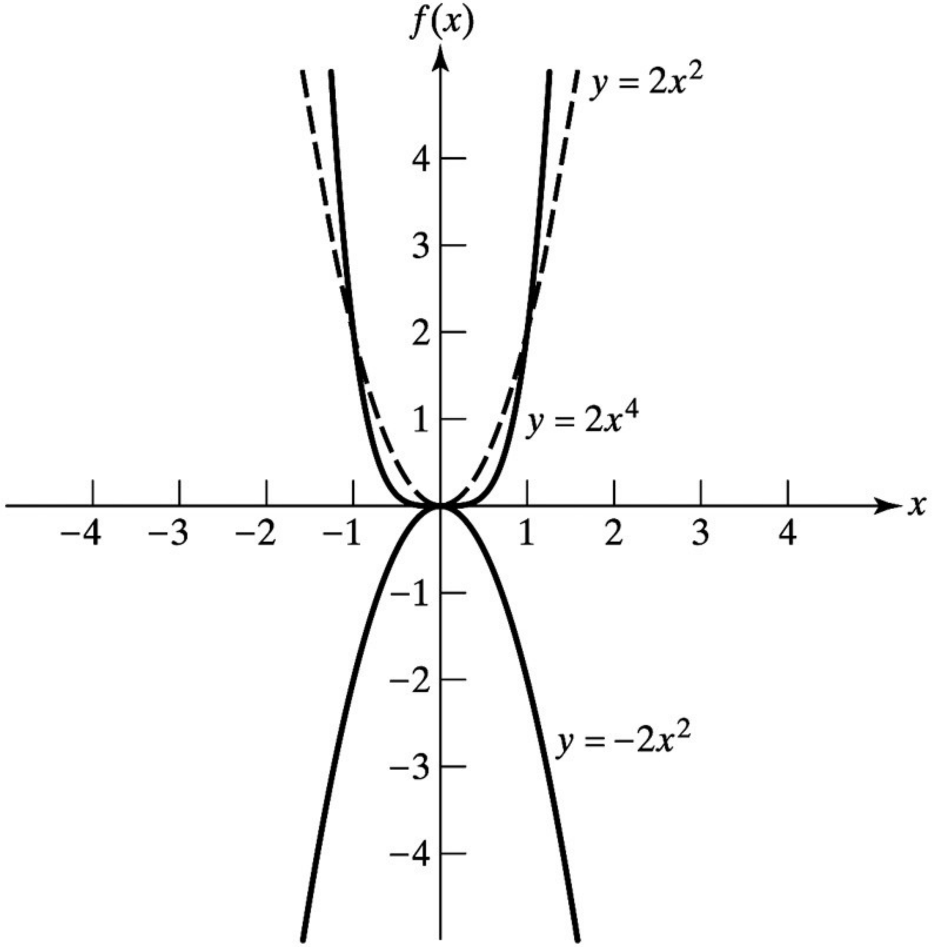
$$x^a y^a = (xy)^a$$

$$\frac{x^a}{y^a} = \left(\frac{x}{y}\right)^a$$

Power functions with p equal even integer

- F(0)=0
- If k>0, then f(x) reaches a global minimum at x=0. If k<0, then f(x) reaches global maximum at x=0.
- These functions are symmetric about the vertical axis. They are strictly convex if k>0 or strictly concave if k<0.

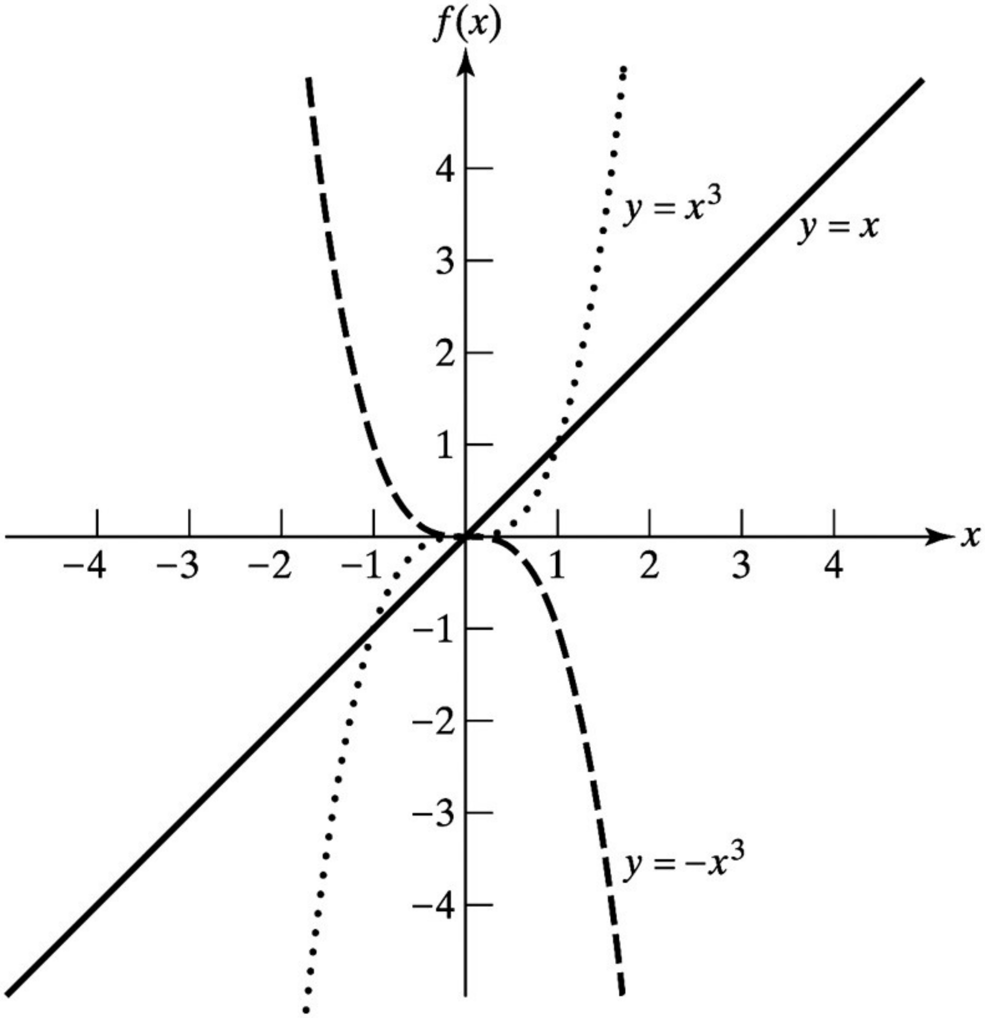
Figure 2.13 Power Functions in Which the Exponent Is a Positive Even Number



Power function when p is positive odd integer

- If $k>0$, then the function is monotonic and increasing. If $k<0$, then the function is monotonic and decreasing.
- If $p \neq 1$ and $k>0$, the function is strictly concave for $x<0$ and strictly convex for $x>0$.
- If $p \neq 1$ and $k<0$, the function is strictly convex for $x<0$ and strictly concave for $x<0$.

Figure 2.14 Power Functions in Which the Exponent Is a Positive Odd Integer



Power function with p negative integer

- The function is non-monotonic
- The function is not continuous and has a vertical asymptote at x=0.

Power function with p negative integer

- If $k > 0$ and p is a negative even integer, then
$$\lim_{x \to 0^-} kx^p = +\infty \ \ and \ \ \lim_{x \to 0^+} kx^p = +\infty$$
- If $k < 0$ and p is a negative even integer, then
$$\lim_{x \to 0^-} kx^p = -\infty \ \ and \ \ \lim_{x \to 0^+} kx^p = -\infty$$
- If $k > 0$ and p is a negative odd integer, then
$$\lim_{x \to 0^-} kx^p = -\infty \ \ and \ \ \lim_{x \to 0^+} kx^p = +\infty$$

Power function with p negative integer

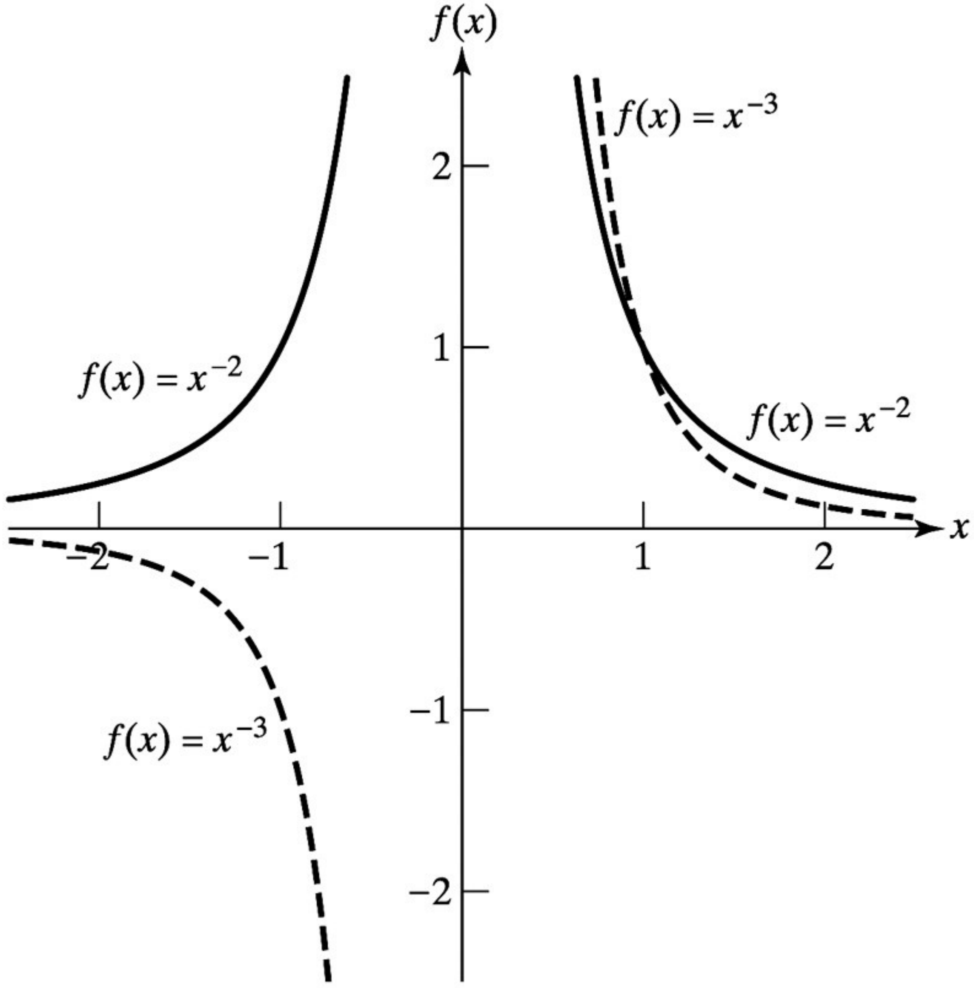
¿If $k<0$ and p is a negative odd integer, then

$$\lim_{x\to 0^-} kx^p = +\infty \quad and \quad \lim_{x\to 0^+} kx^p = -\infty$$

¿If $k>0$ and p is a negative even integer, then kx^p is strictly convex for $x>0$ or for $x<0$.

¿If $k>0$ and p is a negative odd integer, then kx^p is strictly convext for $x>0$ and strictly concave for $x<0$.

Figure 2.15 Power Functions in Which the Exponent Is a Negative Integer



Polynomial Functions

A univariate polynomial function takes the form

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

the parameters $a_i, i = 0,1,2,\cdots,n$ are real numbers and the exponents in the polynomial are integers from 1 to n.

Polynomial Functions

- The **degree** of the polynomial is the value taken by the highest exponent.
- A linear function is polynomial of degree 1.
- A polynomial of degree 2 is called a **quadratic** function.
- A polynomial of degree 3 is called a cubic function.

Roots of Polynomial Function

The roots of a polynomial are the values of its argument that make the function equal zero.

Linear function: $y = a + bx$

root: $x = -a/b$

Quadratic function: $y = ax^2 + bx + c$

roots: $x_1, x_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3 cases for roots of quadratic function

- $b^2-4ac>0$, two distinct roots.
- $b^2-4ac=0$, two equal roots
- $b^2-4ac<0$, two complex roots

Quadratic example

$$y = 2x^2 - 7x + 3$$

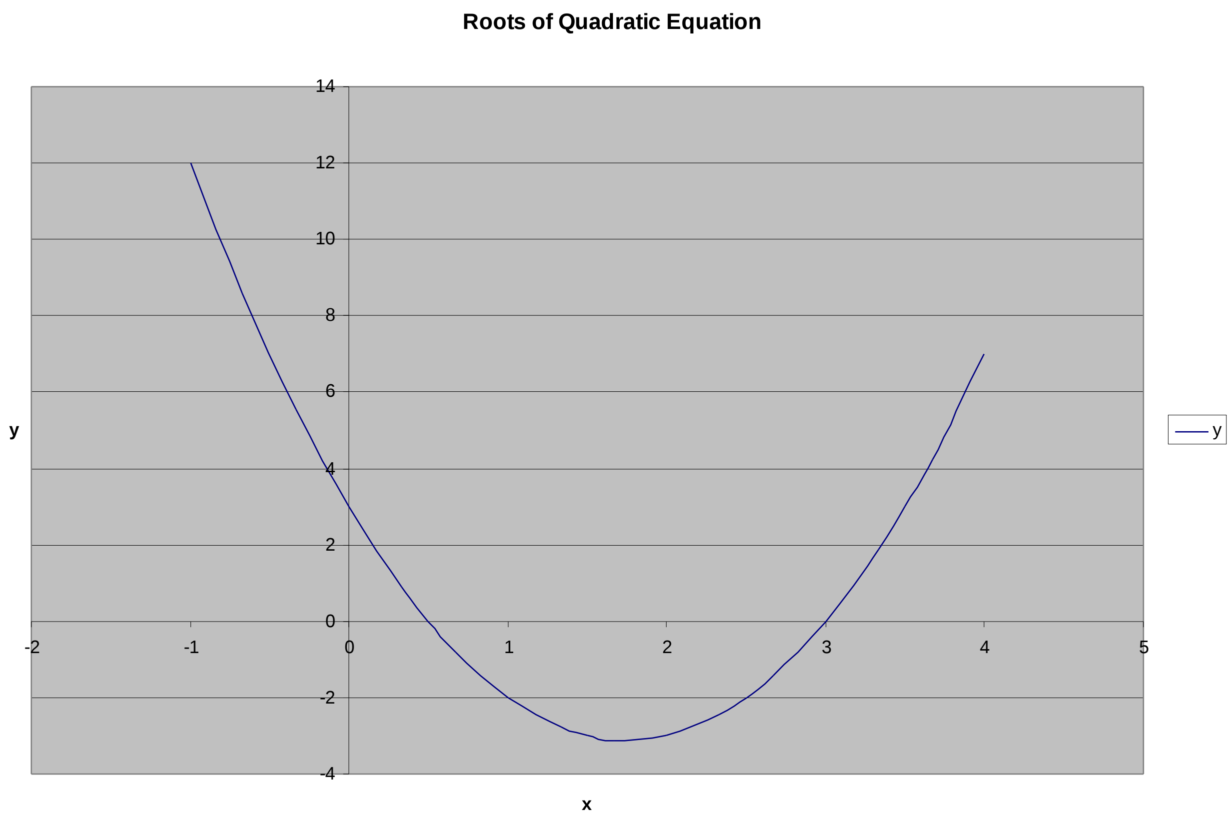
$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{7 - \sqrt{7^2 - 4*3*2}}{2*2}$$

¿ $\frac{7 - \sqrt{49 - 24}}{4} = \frac{7 - \sqrt{25}}{4} = \frac{7 - 5}{4} = \frac{2}{4} = \frac{1}{2}$

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{7 + \sqrt{7^2 - 4*3*2}}{2*2}$$

¿ $\frac{7 + \sqrt{49 - 24}}{4} = \frac{7 + \sqrt{25}}{4} = \frac{7 + 5}{4} = \frac{12}{4} = 3$

Plot of our Quadratic function



Exponential Functions

- The argument of an **exponential function** appears as an exponent.
- $Y = f(x) = kb^x$
- k is a constant and b, called the **base**, is a positive number.
- $f(0) = kb^0 = k$

Exponential Functions

¿ The sign of this function is the same as the sign of the parameter k for any value of x.

¿ When $b>1$, then b^x increases with x. In this case

$$\lim_{x\to-\infty} kb^x = 0$$

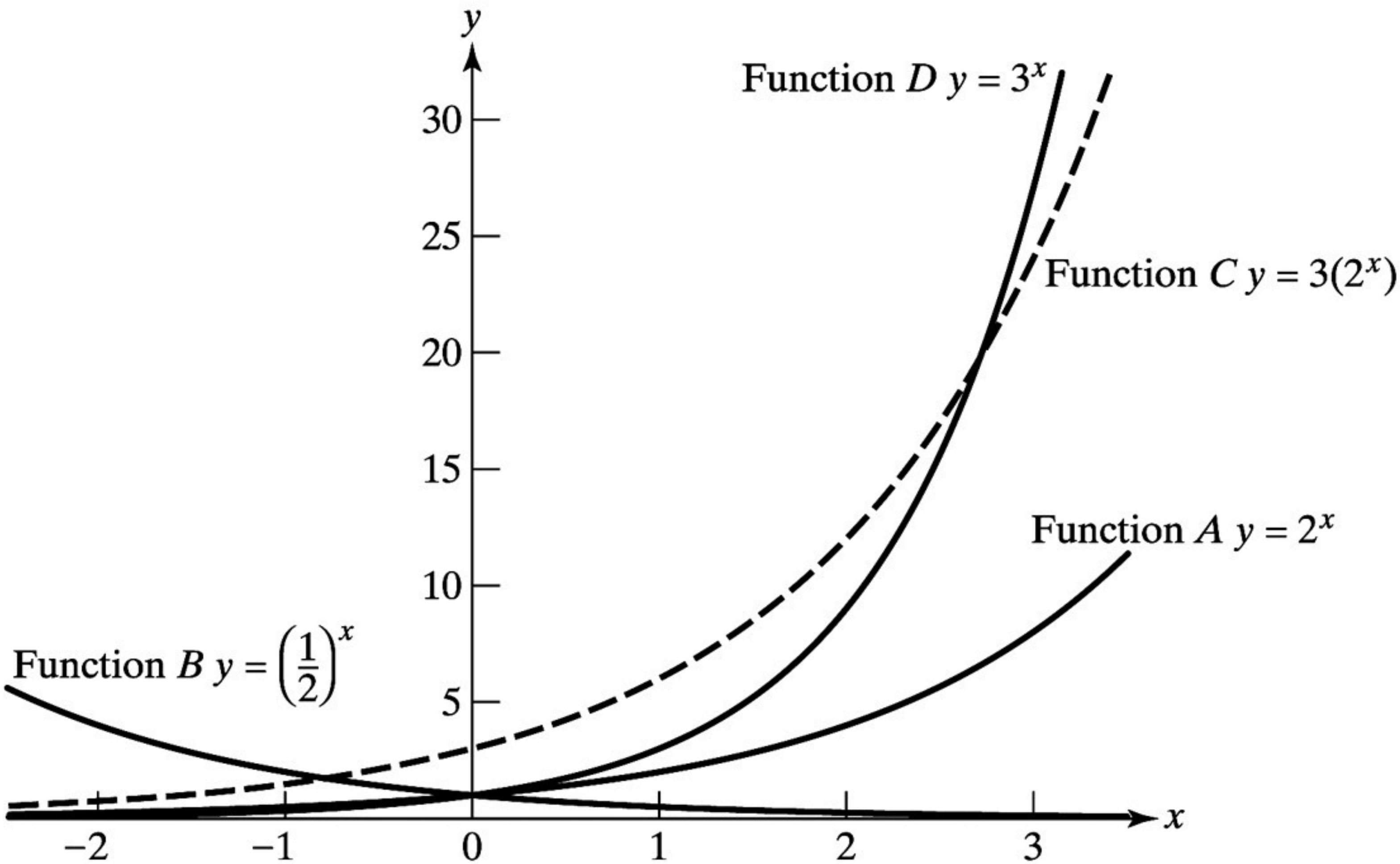
Using our rules of exponents

$$\frac{1}{b^x} = \left(\frac{1}{b}\right)^x = b^{-x}$$ Thus the graph of $\left(\frac{1}{b}\right)^x$ is a reflection of the function b^x across the y-axis.

¿ When $1>b>0$, kb^x montonically decreases with x and

$$\lim_{x\to\infty} kb^x = 0$$

Figure 2.16 Some Exponential Functions



Exponential functions with k<0