Economics 2301

Lecture 11
Matrix Algebra

Acknowledgement

- Much of the material on these slides was taken from Krishnan Namboodiri's book, *MATRIX ALGEBRA: An Introduction,* Book 28, Sage University Paper, 1984.

Arrays

- We see numbers in arrays every day
  - Golf rounds scores by player
  - Weather temperature high and low by city
  - Stock prices by company open, high, low, close
- We now define a “matrix” as a rectangular array of numbers. Our intention is to treat such arrays as single object. To explicitly indicate this intention, we enclose the array with brackets as shown on the next slide.

Matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Matrix Terminology

- The numbers that constitute a matrix are called the *elements entries* of the matrix.
- We refer to the elements by their row and column numbers. Thus, the (2,1) element of the matrix on the previous slide is 4; the (2,3) element of the matrix is 6; and so on. Obviously, if a matrix has n rows and m columns, it has altogether nm elements.
- A matrix that has n rows and m columns is said to be of order n by m or nXm.
- In giving the order of a matrix, we always mention the number of rows first, follwed by the number of columns.

Matrix Conventions

- Use single letter as a label for a matrix.
- Use letters to designate a matrix's elements.
- Use Bold capital letters for matrices.
- Use lower-case, ordinary letters for their elements.

Examples of Matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}$$

General Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\mathrm{n}} \\ a_{21} & a_{22} & \cdots & a_{2\mathrm{n}} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

More Matrix Terminology

- Matrix containing only one row is called a *row vector*.
- Matrix containing only one column is called a *column vector.*
- $\begin{bmatrix} a & b & c \end{bmatrix}$ *is a* row vector.
- $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *is a* column vector.

Equality of Matrices

- Two matrices are equal if (a) they both have the same number of rows and the same number of columns, and (b) their corresponding elements are equal.
- In symbols, if $A_{nXm} = ((a_{ij}))$ and $B_{rXs} = ((b_{ij}))$ the A=B, i.e., A and B are equal if $n=r$ and $m=s$ and $a_{ij}=b_{ij}$ for $i = 1,2,...,n(=r)$; $j=1,2,...,m(=s)$.

Example of Equality

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & x \\ y & 4 \end{bmatrix}$

then A = B implies that x=2 and y=3.

Addition and Subtraction of Matrices

- We define addition (subtraction) of matrices in terms terms of addition (subtraction) of their corresponding elements. The sum of two *n* by *m* matrices is an *n* by *m* matrix whose elements are the sum of the corresponding elements of the original matrices.
- In symbols, if $A_{nXm}=((a_{ij}))$ and $B_{nXm}=((b_{ij}))$, then their sum, denoted by A+B, is $((a_{ij}+b_{ij}))$.
- Note that we shall add two matrices only if they are of the same order.

Addition of Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$A + B = \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) \\ (a_{21} + b_{21}) & (a_{22} + b_{22}) \end{bmatrix}$$

Addition of Matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1+5) & (2+6) \\ (3+7) & (4+8) \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Subtraction of Matrices

$$A - B = ((a_{ij} - b_{ij}))$$

Thus, if $A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ then

$$A - B = \begin{bmatrix} -2-1 & 0-(-1) & 1-0 \\ 0-1 & 1-0 & -1-(-1) \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Multiplication by a Scalar

- Let k be an ordinary number (scalar) and $A=((a_{ij}))$ be any matrix.
- $kA=((ka_{ij}))$
- To multiple a matrix by an ordinary number (scalar), we multiply each element of the matrix by the number.
- thus, 2Q=Q+Q, where Q is any matrix.

Scalar multiplication

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2*2 & 2*3 \\ 2*1 & 2*4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 8 \end{bmatrix}$$

$$A + A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 8 \end{bmatrix} = 2A$$

Vectors

- We shall denote a vector by a boldface, lower-case letter, and refer to a vector consisting of *n* elements by the term *n*-tuple.
- Two column (row) vectors are said to be equal if they have the same number of elements and their corresponding elements are equal.
- A row vector is never equal to a column vector.
- Equality of two vectors **a** and **b** is denoted by **a**=**b**.

Vectors Continued

- The sum of two column vectors with the same number of elements (or of two row vectors with the same number of elements) is formed by adding the corresponding elements of the given vectors.
- The sum of two vectors **a** and **b** is denoted by **a**+**b**.
- Let **a** be a column or row-vector and *k* an ordinary number. Then the product *k***a** is defined as the vector whose elements are *k* times the corresponding elements of **a**.

Examples

$$\begin{bmatrix} 9 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2*4 \\ 2*2 \\ 2*5 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}$$

Vector Representation of a system of Linear Equations

The vector operations previously defined can be used to express a system of linear equations compactly as a single vector equation. Consider, for example, the following two equations in two unknowns:

$$2x + 3y = 5$$
$$3x + 2y = 5$$

Let us form the following three column vectors, corresponding to the coefficients of x, those of y, and the constant terms:

$$\boldsymbol{a} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{and } \boldsymbol{c} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Vector Representation Cont.

Now the given set of equations can be expressed compactly as

$$xa + yb = c$$

To verify, note

$$x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

which, by virtue of the definition of scalar multiplication, becomes

$$\begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} 3y \\ 2y \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

which, in turn, by virtue of the definition of addition is the same as

$$\begin{bmatrix} (2x + 3y) \\ (3x + 2y) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

and by definition of equality, we have

2x + 3y =5

3x + 2y = 5

Inner Product

Let **a**' be a row vector and **b** a column vector, both being *n*-tuples, that is vectors having *n* elements:

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$a' = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$$

then the product **a**' times **b** is defined to be the scalar $a_1b_1 + ... + a_nb_n$. This product is denoted **a'b** or $a' \cdot b$. It is sometimes called the *inner product* or *dot product* of **a**' and **b**.

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = (1)(2) + (2)(4) + (4)(3) = 22.$$

Transpose

- We say that [2 3] and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are *transposes* of each other. More generally,
- $\begin{bmatrix} a_1 & \dots & a_n \end{bmatrix}$ is the transpose of $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.
- The usual notation for the transpose of **a** is **a'** or $\mathbf{a}^T$.
- It is easy to see that the transpose of a transpose of a vector is the original vector. In symbols, **(a')'=a**.

Transpose of a matrix

- If **A** is an *nXm* matrix, the the *mXn* matrix **A'** obtained by interchanging the rows and columns of **A** is called the transpose of A.

- For example, $\begin{bmatrix} 3 & 8 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 \\ 8 & 1 \end{bmatrix}$ are transposes of each other.

1^{st} Key Inner Product

Sum of Squares

$$(x_1)^2 + \ldots + (x_n)^2 = x'x$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $x' = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$

2^{nd} Key Inner Product

Sum of Cross Products

$$x'y = y'x = (x_1 y_1) + \cdots + (x_n y_n)$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

Matrix-Vector Multiplication

Let **A** be a matrix and **v** a column vector such that the number of columns of **A** equals the number of elements in **v**. Then the product **A** times **v**, written **Av**, is a column vector **c** whose i^{th} element is equal to the inner product of the i^{th} row of **A** with **v**.

Example of Matrix-vector multiplication

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix} = c$$

the first element of **c** is $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix}, i.e., 25.$

The fourth element is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix}, i.e., 40.$

Example continued

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix} = \begin{bmatrix} 25 \\ 35 \\ 39 \\ 40 \end{bmatrix}$$

Matrix Multiplication

We define an operation that produces a matrix **C** by concatenating horizontally a given matrix **A** times the successive columns of another matrix **B**. We define such a concatenation involving **A** and **B** the product **A** times **B**, usually denoted **AB**. The operation that produces such a concatenation is called matrix-matrix multiplication or simply matrix multiplication. Using the matrices introduced above, we say that **AB=C** stipulating that, as mentioned above, **AB** means the horizontal concatenation in which **A** times the first column of **B** is followed on the right by **A** times the second column of **B**.

Notice that this operation (i.e., matrix multiplication as defined above) applies only if the number of columns in the left-factor (**A** in our example) equals the number of rows in the right-fact (**B** in the example).

Matrix Multiplication Example

$$E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad F = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$EF = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} \right] = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$FE = \left[\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Alternative Definition of Matrix Multiplication

An alternative way of defining matrix multiplication is the following: Given $\mathbf{A}_{nXm} = ((a_{ij}))$ and $\mathbf{B}_{mXp}=((b_{ij}))$, the product **AB** is an *(nXp)* matrix whose *(i,j)* element equals the inner product of the i^{th} row of **A** with the j^{th} column of **B**.

Example

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\7 & 8\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix} & \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}6\\8\end{bmatrix}\\ \begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix} & \begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}6\\8\end{bmatrix}\end{bmatrix}=\begin{bmatrix}19 & 22\\43 & 50\end{bmatrix}$$