Economics 2301

Matrices
Lecture 12

Inner Product

Let **a**' be a row vector and **b** a column vector, both being *n*-tuples, that is vectors having *n* elements:

$$a' = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

then the product **a**' times **b** is defined to be the scalar $a_1 b_1 + \ldots + a_n b_n$. This product is denoted **a'b** or $a' \cdot b$. It is sometimes called the *inner product* or *dot product* of **a**' and **b**.

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = (1)(2) + (2)(4) + (4)(3) = 22.$$

Transpose

- We say that [2 3] and $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ are *transposes* of each other. More generally,
- $\begin{bmatrix} a_1 & \ldots & a_n \end{bmatrix}$ is the transpose of $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$
- The usual notation for the transpose of **a** is **a'** or $\mathbf{a}^{\mathbf{T}}$.
- It is easy to see that the transpose of a transpose of a vector is the original vector. In symbols, **(a')'=a**.

Transpose of a matrix

- If **A** is an *nXm* matrix, the the *mXn* matrix **A**' obtained by interchanging the rows and columns of **A** is called the transpose of A.

- For example, $\begin{bmatrix} 3 & 8 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 4 \\ 8 & 1 \end{bmatrix}$ are transposes of each other.

1^{st} Key Inner Product

Sum of Squares

$$(x_1)^2 + \ldots + (x_n)^2 = x'\,x$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $x' = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}$

2nd Key Inner Product

Sum of Cross Products

$$x'y = y'x = (x_1 y_1) + \cdots + (x_n y_n)$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

Matrix-Vector Multiplication

Let **A** be a matrix and **v** a column vector such that the number of columns of **A** equals the number of elements in **v**. Then the product **A** times **v**, written **Av**, is a column vector **c** whose i^{th} element is equal to the inner product of the i^{th} row of **A** with **v**.

Example of Matrix-vector multiplication

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix} = c$$

the first element of **c** is $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix}, i.e., 25.$

The fourth element is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix}, i.e., 40.$

Example continued

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \end{bmatrix} = \begin{bmatrix} 25 \\ 35 \\ 39 \\ 40 \end{bmatrix}$$

Matrix Multiplication

We define an operation that produces a matrix **C** by concatenating horizontally a given matrix **A** times the successive columns of another matrix **B**. We define such a concatenation involving **A** and **B** the product **A** times **B**, usually denoted **AB**. The operation that produces such a concatenation is called matrix-matrix multiplication or simply matrix multiplication. Using the matrices introduced above, we say that **AB=C** stipulating that, as mentioned above, **AB** means the horizontal concatenation in which **A** times the first column of **B** is followed on the right by **A** times the second column of **B**.

Notice that this operation (i.e., matrix multiplication as defined above) applies only if the number of columns in the left-factor (**A** in our example) equals the number of rows in the right-fact (**B** in the example).

Matrix Multiplication Example

$$E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad F = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$EF = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} \right] = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$FE = \left[\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Alternative Definition of Matrix Multiplication

An alternative way of defining matrix multiplication is the following: Given $\mathbf{A}_{nXm} = ((a_{ij}))$ and $\mathbf{B}_{mXp}=((b_{ij}))$, the product **AB** is an *(nXp)* matrix whose *(i,j)* element equals the inner product of the i^{th} row of **A** with the j^{th} column of **B**.

Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} & \begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Determinant

a) Let us stipulate that the determinant of a (1x1) matrix is the numerical value of the sole element of the matrix.

b) For a 2x2 matrix **A** (given below), we will define the determinant of **A**, noted det(**A**) or |**A**|, to be ad-bc.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Cofactor

The cofactor of the (i,j) element of **A** may be defined as $(-1)^{i+j}$ times the determinant of the submatrix formed by omitting the i^{th} row and the j^{th} column of **A**. We can put these cofactors in a matrix we call the cofactor matrix. Let **W** be the cofactor matrix of our 2x2 matrix A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$W = \begin{bmatrix} (-1)^{1+1} d & (-1)^{(1+2)} c \\ (-1)^{(2+1)} b & (-1)^{(2+2)} a \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Determinant Continued

It is possible to define the determinant of a (3x3) matrix in terms of determinants of (2x2) matrices as the weighted sum of the elements of any row or column of the given (3x3) matrix, using as weights the respective cofactors – the cofactor of the (i,j) element of the (3x3) matrix being $(-1)^{i+j}$ times the determinant of the (2x2) submatrix formed by omitting the i^{th} row and the j^{th} column of the original matrix. This definition is readily generalizable. Let

$$F = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

The cofactor matrix for F, G, is

$$G = \begin{bmatrix} (-1)^{(1+1)}\begin{vmatrix} 7 & 2 \\ 1 & 1 \end{vmatrix} & (-1)^{(1+2)}\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} & (-1)^{(1+3)}\begin{vmatrix} 4 & 7 \\ 3 & 1 \end{vmatrix} \\ (-1)^{(2+1)}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & (-1)^{(2+2)}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & (-1)^{(2+3)}\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \\ (-1)^{(3+1)}\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & (-1)^{(3+2)}\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} & (-1)^{(3+3)}\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -17 \\ -2 & -1 & 7 \\ -1 & 0 & 0 \end{bmatrix}$$

Determinant Continued

We can get the determinant of **F** by expanding on the first row

$$|F| = 2*5+3*2+1*(-17) = 10+6-17 = -1$$

We also get the same value for the determinant of **F** if we expand on the first column of **F**.

$$|F| = 2*5+4*(-2)+3*(-1) = 10-8-3 = -1$$

Prove for yourself that you get the same value for the determinant of **F** **i**f you expand on any other row or column.

General Rule for Determinant

Minor: Let **A** be an *nxn* matrix. Let $\mathbf{M}_{ij}$ be the (n-1)x(n-1) matrix obtained by deleting the i^{th} row and the j^{th} column of **A**. The determinant of that matrix, denoted $|\mathbf{M}_{ij}|$, is called the minor of a_{ij}.

Cofactor: Let A be an *nxn* matrix. The cofactor C_{ij} is a minor multiplied by $(-1)^{(i+j)}$. That is, $C_{ij}=(-1)^{(i+j)}|M_{ij}|$

Laplace Expansion: The general rule for finding the determinant of an *nxn* matrix **A**, with the representative element a_{ij} and the set of cofactors C_{ij}, through a Laplace expansion along its i^{th} row, is

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$$

The same determinant can be evaluated by a Laplace expansion along the j^{th} column of the matrix, which give us

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$$