Economics 2301

Matrices

Lecture 13

Determinant

a) Let us stipulate that the determinant of a (1x1) matrix is the numerical value of the sole element of the matrix.

b) For a 2x2 matrix **A** (given below), we will define the determinant of **A**, noted det(**A**) or |**A**|, to be ad-bc.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Cofactor

The cofactor of the (i,j) element of **A** may be defined as $(-1)^{i+j}$ times the determinant of the submatrix formed by omitting the i^{th} row and the j^{th} column of **A**. We can put these cofactors in a matrix we call the cofactor matrix. Let **W** be the cofactor matrix of our 2x2 matrix A.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$W = \begin{bmatrix} (-1)^{1+1} d & (-1)^{(1+2)} c \\ (-1)^{(2+1)} b & (-1)^{(2+2)} a \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Determinant Continued

It is possible to define the determinant of a (3x3) matrix in terms of determinants of (2x2) matrices as the weighted sum of the elements of any row or column of the given (3x3) matrix, using as weights the respective cofactors – the cofactor of the (i,j) element of the (3x3) matrix being $(-1)^{i+j}$ times the determinant of the (2x2) submatrix formed by omitting the i^{th} row and the j^{th} column of the original matrix. This definition is readily generalizable. Let

$$F = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

The cofactor matrix for F, G, is

$$G = \begin{bmatrix} (-1)^{(1+1)}\begin{vmatrix} 7 & 2 \\ 1 & 1 \end{vmatrix} & (-1)^{(1+2)}\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} & (-1)^{(1+3)}\begin{vmatrix} 4 & 7 \\ 3 & 1 \end{vmatrix} \\ (-1)^{(2+1)}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & (-1)^{(2+2)}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & (-1)^{(2+3)}\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \\ (-1)^{(3+1)}\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & (-1)^{(3+2)}\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} & (-1)^{(3+3)}\begin{vmatrix} 2 & 3 \\ 4 & 7 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 2 & -17 \\ -2 & -1 & 7 \\ -1 & 0 & 2 \end{bmatrix}$$

Determinant Continued

We can get the determinant of **F** by expanding on the first row

$$|F| = 2 * 5 + 3 * 2 + 1 * (-17) = 10 + 6 - 17 = -1$$

We also get the same value for the determinant of **F** if we expand on the first column of **F**.

$$|F| = 2 * 5 + 4 * (-2) + 3 * (-1) = 10 - 8 - 3 = -1$$

Prove for yourself that you get the same value for the determinant of **F** **i**f you expand on any other row or column.

General Rule for Determinant

Minor: Let **A** be an *nxn* matrix. Let $\mathbf{M}_{ij}$ be the (n-1)x(n-1) matrix obtained by deleting the i^{th} row and the j^{th} column of **A**. The determinant of that matrix, denoted $|\mathbf{M}_{ij}|$, is called the minor of a_{ij}.

Cofactor: Let A be an *nxn* matrix. The cofactor C_{ij} is a minor multiplied by $(-1)^{(i+j)}$. That is, $C_{ij}=(-1)^{(i+j)}|M_{ij}|$

Laplace Expansion: The general rule for finding the determinant of an *nxn* matrix **A**, with the representative element a_{ij} and the set of cofactors C_{ij}, through a Laplace expansion along its i^{th} row, is

$$|A| = \sum_{j=1}^{n} a_{ij} C_{ij}$$

The same determinant can be evaluated by a Laplace expansion along the j^{th} column of the matrix, which give us

$$|A| = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Interesting result

We discovered that if we expanded by either a row or column, summing up the product of the elements of the row (column) by the cofactors of that row (column) that we got the determinant of the matrix (nxn). Now if we take the matrix of cofactors and transpose it we get a matrix known as the adjoint matrix.

Let $\mathbf{M}_{nxn}$ = ((m_{ij})) is any (nxn) matrix and $\mathbf{C}_{nxn}$ = ((c_{ij})) is such that c_{ij} is the cofactor of m_{ij} (for i-1,...,n; j=1,...,n), then

$$\mathbf{MC' = C'M = |M|I_n}$$

Where I is an *nxn* diagonal matrix with 1s down the diagonal and zeros of the diagonal. It is known as the identity matrix.

C' is known as the adjoint matrix. I.e. the adjoint matrix is the transpose of the cofactor matrix.

Note in the first multiplication that we are getting expansion by rows and in the second multiplication by columns. The off-diagonal zeros are due to a rule known as expansion by alien cofactors.

Identity Matrix

In ordinary algebra we have the number 1, which has the property that its product with any number is the number itself. In matrix algebra, the corresponding matrix is the Identity Matrix. It is a square matrix -one having the same number of rows and columns – and it has unity in the principal diagonal (i.e., the diagonal of elements from the upper left corner to the lower right corner) and 0 everywhere else. It is usually labeled, $\mathbf{I_n}$ for an *nxn* matrix or simply **I.** It has the property for any matrix, A, that is conformable for multiplication that **IA=AI=A**.

$$Let A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1(5)+0(3) & 1(2)+0(4) \\ 0(5)+1(3) & 0(2)+1(4) \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$$AI = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5(1)+2(0) & 5(0)+2(1) \\ 3(1)+4(0) & 3(0)+4(1) \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} = A$$

Inverse Matrix

In arithmetic and ordinary algebra there is an operation of division. Can we define an analogous operation for matrices? Strictly speaking, there is no such thing as division of one matrix by another; but there is an operation that accomplishes the same thing as division does in arithmetic and scalar algebra.

In arithmetic, we know that multiplying by 2^{-1} is the same thing as dividing by 2. More generally, given any nonzero scalar a, we can speak of multiplying by a^{-1} instead of dividing by a. The multiplication by a^{-1} has the property that $aa^{-1} = a^{-1} a = 1$.

This prompts the question, for a matrix **A**, can we find a matrix **B** such that $\mathbf{BA} = \mathbf{AB} = \mathbf{I}_{nxn}$ where **I** is an identity matrix of order n (the matrix analogue of unity).

In order for this to hold, **AB** and **BA** must be of order nXn; but **AB** is of order nXn only if **A** has n rows and **B** has n columns, and **BA** is of order nXn only if **B** has n rows and **A** has n columns. Therefore the above only holds if **A** and **B** are both of order nXn. This leads to the following definition:

Given a square matrix **A**, if there exists a square matrix **B**, such that

$$\mathbf{BA} = \mathbf{AB} = \mathbf{I}$$

then **B** is called the inverse matrix (or simply the inverse) of **A**, and **A** is said to be invertible. Not all square matrices are invertible. We label the matrix **B** as $\mathbf{A}^{-1}$.

Example

Given a matrix A,

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$$

$$we\ \ find\ that$$

$$B = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$$

$$satsifies\ the\ relations\ AB = BA = I.$$

$$AB = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1(4)+1(-3) & 1(-1)+1(1) \\ 3(4)+4(-3) & 3(-1)+4(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4(1)+-1(3) & 4(1)+-1(4) \\ -3(1)+1(3) & -3(1)+1(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Properties of Inverse Matrices

Property 1: For any nonsingular matrix **A**. $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

Property 2: The inverse of a matrix **A** is unique.

Property 3: For any nonsingular matrix **A**, $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$.

Property 4: If **A** and **B** are nonsingular and of the same dimension, then **AB** is nonsingular and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Inverse Matrix

We have the interesting result that if if $\mathbf{M}_{nXn} = ((m_{ij}))$ is any (*nXn*) matrix and $\mathbf{C} = ((c_{ij}))$ is such that c_{ij} is the cofactor of m_{ij} (for i-1,...,n; j=1,...n), then

$$\mathbf{MC' = C'M = |M|I_n}.$$

This implies, among other things (see below), that if we multiply each element of **C'** (the adjoint matrix) by the reciprocal of **|M|**, provided, of course, $|\mathbf{M}| \neq 0$ the resulting matrix is $\mathbf{M}^{-1}$.

$$\mathbf{M^{-1}} = (1/|\mathbf{M}|)\mathbf{C}'.$$

Inverse Example

Consider the matrix F from slide 4 with cofactor matrix **G**. The |F| =-1 (slide 5).

$$F = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 3 & 1 & 1 \end{bmatrix} \quad G = \begin{bmatrix} 5 & 2 & -17 \\ -2 & -1 & 7 \\ -1 & 0 & 2 \end{bmatrix}$$

The inverse matrix is (1/|**F**|)**G**'

$$F^{-1} = (1/-1)G' = \begin{bmatrix} -5 & 2 & 1 \\ -2 & 1 & 0 \\ 17 & -7 & -2 \end{bmatrix}$$

$$F^{-1}F = \begin{bmatrix} -5 & 2 & 1 \\ -2 & 1 & 0 \\ 17 & -7 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 7 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -5(2)+2(4)+1(3) & -5(3)+2(7)+1(1) & -5(1)+2(2)+1(1) \\ -2(2)+1(4)+0(3) & -2(3)+1(7)+0(1) & -2(1)+1(2)+0(1) \\ 17(2)+-7(4)+-2(3) & 17(3)+-7(7)+-2(1) & 17(1)+-7(2)+-2(1) \end{bmatrix}$$

$$F^{-1}F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Solving Equation Systems

Write a general equation system as :

$$a_{11}x_1 + \ldots + a_{1n}x_n = c_1$$

$$\vdots$$

$$a_{n1}x_1 + \ldots + a_{nn}x_n = c_n$$

$$let\ A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

We can write the sytem compactly as $Ax = c.$

The solution for the system is then $x = A^{-1}c$

Example

suppose we had the equation system:

$$3x + y = 7$$

$$x + y = 3$$

$$Here\ A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad c = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$|A| = 2 \quad Cofactor(A) = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$1/(|A|)Cofactor(A)' = (1/2)\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = A^{-1}$$

Example continued

$$x = A^{-1}c = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} (1/2)7 + (-1/2)3 \\ (-1/2)7 + (3/2)3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Cramer's Rule

Cramer's Rule: For the system of equations **Ax** = **y,** where A is an nxn nonsingular matrix, the solution for the ith endogenous variable, x_i, is

$$\mathbf{x_i = |A_i|/|A|}$$

where the matrix $\mathbf{A}_i$ represents a matrix that is identical to the matrix **A** but for the replacement of the *i*th column with the *nx1* vector **y**.

Our Example – Cramer's Rule

$$\text{Here } A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \quad c = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$x = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 7 & 1 \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 3 & 1 \\ 7 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}} = \frac{2}{2} = 1$$

The same solution we got earlier.