Economics 2301

Lecture 15

Differential Calculus

Difference Quotient

The difference quotient, $\frac{\Delta y}{\Delta x}$, shows how the value of a function, $y = f(x)$, changes with respect to a change in its argument.

Example excise tax revenue

Case 1: Perfectly inelastic demand

$$R = t\overline{Q} \qquad \Delta R = R_i - R_j = t_i\overline{Q} - t_j\overline{Q} = (t_i - t_j)\overline{Q}$$

$$= \Delta t \cdot \overline{Q}$$

Difference Quotient

$$\frac{\Delta R}{\Delta t} = \frac{\Delta t \cdot \overline{Q}}{\Delta t} = \overline{Q}$$

Example excise tax revenue

Case 2 : Unitary Elasticity Demand

$$R = \frac{tC}{P+t} \quad \text{where } P = \text{supply price, } t = \text{tax rate and } C = \text{constant.}$$

$$\Delta R = \frac{(t+\Delta t)C}{(P+t+\Delta t)} - \frac{tC}{P+t}$$

$$= \frac{(P+t)(t+\Delta t)C - (P+t+\Delta t)tC}{(P+t+\Delta t)(P+t)}$$

$$= \frac{C\cdot\left[Pt + P\Delta t + t^2 + t\Delta t - Pt - t^2 - t\Delta t\right]}{(P+t+\Delta t)(P+t)} = \frac{C\cdot[P\Delta t]}{(P+t+\Delta t)(P+t)}$$

$$\frac{\Delta R}{\Delta t} = \frac{P\cdot C}{(P+t)^2 + \Delta t\cdot(P+t)}$$

Excise Tax Example Continued

Tax Rate	R=30t		R=100t/(2+t)	
	R	ΔR/Δt	R	ΔR/Δt
1	30		33.33	
		30		17.67
2	60		50	
		30		10
3	90		60	

General Form of Difference Quotient

In general

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Note that in general

$$f(x+\Delta x) \neq f(x) + f(\Delta x)$$

Example

$f(x) = 4 + 5/x$ for $x = 5$ and $\Delta x = 1$

$$f(x+\Delta x) = f(6) = 4 + 5/6 = 4.8333$$

$$f(x) + f(\Delta x) = f(5) + f(1) = (4+5/5) + (4+5/1)$$

$$= 5 + 9 = 14 \neq 4.8333$$

Difference quotient for linear and quadratic functions

$$\text{Linear function}: y = a + bx$$

$$\frac{\Delta y}{\Delta x} = \frac{a + b(x + \Delta x) - a - bx}{\Delta x} = \frac{b\Delta x}{\Delta x} = b$$

$$\text{Quadratic function}: y = a + bx + cx^2$$

$$\frac{\Delta y}{\Delta x} = \frac{a + b(x + \Delta x) + c(x + \Delta x)^2 - a - bx + cx^2}{\Delta x}$$

$$= \frac{a + bx + b\Delta x + cx^2 + 2cx\Delta x + c(\Delta x)^2 - a - bx + cx^2}{\Delta x}$$

$$= b + 2cx + c\Delta x$$

Change at the margin and the derivative

In general, the difference quotient of a function $y=f(x)$ depends upon the parameters of the original function, as well as the argument of the function, x, and the change in the argument of the function, Δx. We can, in a sense, standardize the difference quotient by specifying a common Δx to be used in the calculation of all difference quotients. In calculus, we consider the limit of the difference quotient as Δx approaches zero. We call the difference quotient as Δx approaches zero the **derivative**.

Derivative

The derivative of the function $y = f(x)$, which is written as dy/dx, is

$$dy/dx = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists.

Derivative Linear function

Recall for our linear function $\mathrm{y} = a + bx$ that

$$\frac{\Delta y}{\Delta x} = \frac{f\left(x+\Delta x\right) - f\left(x\right)}{\Delta x} = b$$

$$dy/dx = \lim_{\Delta x \to 0} \frac{f\left(x+\Delta x\right) - f\left(x\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} b = b$$

Derivate of Quadratic function

Recall that for our quadratic function $\mathrm{y = a + bx + cx^2}$

the difference quotient was $b + 2cx + c\Delta x$, therefore

$$dy/dx = \lim_{\Delta x \to 0} \left(b + 2cx + c\Delta x \right) = b + 2cx$$

Derivative of our tax function for constant elasticity demand

Recall that for our constant elasticity demand function,

the difference quotient for the tax revenue function was

$$\frac{\Delta R}{\Delta t} = \frac{P \cdot C}{(P+t)^2 + \Delta t \cdot (P+t)} \quad \textit{hence}$$

$$dR/dt = \lim_{\Delta t \to 0}\left(\frac{P \cdot C}{(P+t)^2 + \Delta t \cdot (P+t)}\right) = \frac{P \cdot C}{(P+t)^2}$$

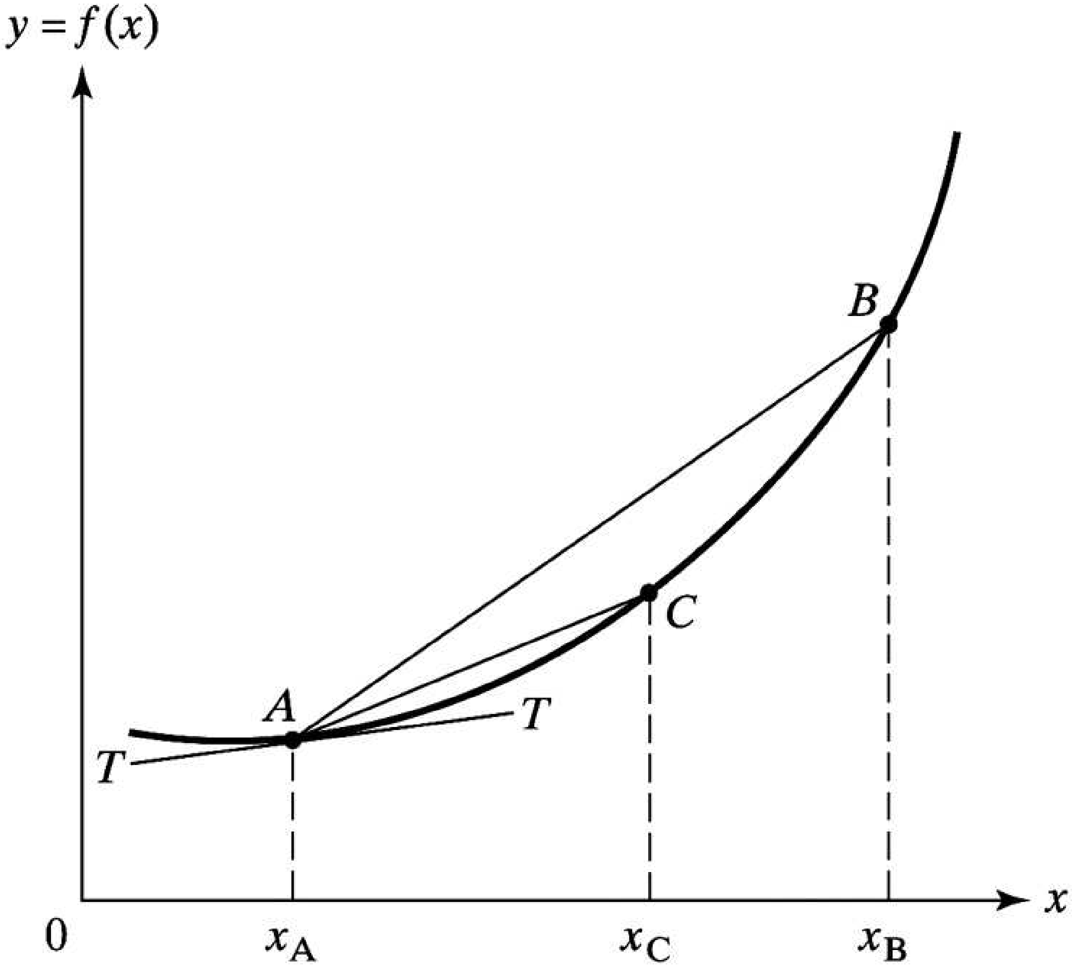
Derivative and Economics

- Since the derivative is defined for very small changes in x, it is some times referred to as the **instantaneous rate of change** of the function.
- The mathematical concept of the derivative has a direct correspondence to the economic concept of looking at relationships "at the margin."
- Economic concepts such as marginal revenue, marginal productivity and marginal propensity to consume.

Geometry of Derivatives

- The difference quotient $\Delta R/\Delta t$ was also referred to as the slop of the secant line.
- The derivative of a function at a given value represents the slope of a line tangent to that function at that value.

Figure 6.2 Secant Lines and a Tangent Line



Derivative as a function

- In many cases, the derivate dy/dx is itself a function of x. i.e. dy/dx=f'(x).

Our constant elasticity tax revenue function

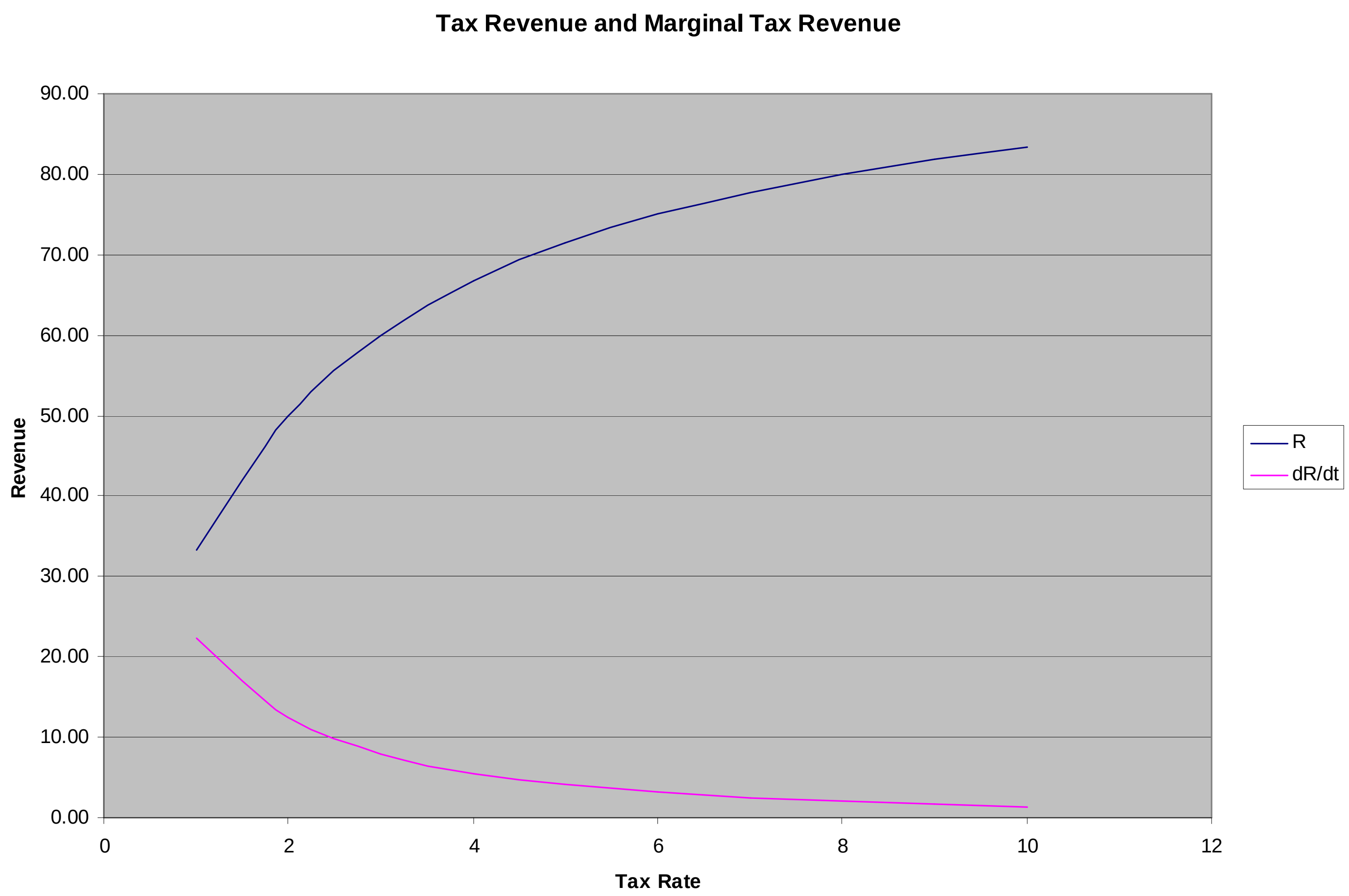
Our tax revenue function for a constant elasticity demand function was

$$R = \frac{tC}{P+t}$$

The derivative of the function was

$$dR/dt = \frac{PC}{(P+t)^2}$$

Plot of Tax Revenue and Marginal Tax Revenue



Geometry of Derivatives

1. The derivative $g'(x)$ is positive when the slope of a line tangent to $g(x)$ is positive regardless of whether $g(x)$ itself is negative or positive.(see range x_a to x_b). $g'(x)$ is negative when the slope of a line tanget to $g(x)$ is negative, as in the range x_b to x_c .

2. The derivative $g'(x)$ equals zero when the slope of a line tangent to $g(x)$ is zero.

3. The derivative of the function equals zero at any local or global extreme point. See $g(x_a)$, $g(x_b)$ and $g(x_c)$.

4. The derivative equals zero at $g(x_d)$ which is an inflection point.

Figure 6.4 A Function and Its Derivative