Economics 2301

Lecture 17

Differential
Average and marginal functions

- The average function rises as long as the marginal function lies above it.
- The average function declines as long as the marginal function lies below it.
Average and marginal functions

If \( f'(x_0) > \frac{f(x_0)}{x_0} \Rightarrow \) for a small change in \( x \) in the neighborhood of \( x_0 \), \( \frac{f(x_0 + \Delta x)}{x_0 + \Delta x} > \frac{f(x_0)}{x_0} \).

Proof:

\( f'(x_0) > \frac{f(x_0)}{x_0} \Rightarrow \) for small \( \Delta x \)

\[ \frac{f(x_0 + \Delta x)}{\Delta x} > \frac{f(x_0)}{x_0} + \frac{f(x_0)}{\Delta x} \quad \text{or} \]

\[ \frac{f(x_0 + \Delta x)}{\Delta x} > \frac{(x_0 + \Delta x)f(x_0)}{x_0 \cdot \Delta x} \quad \text{or} \]

\[ \frac{\Delta x}{x_0 + \Delta x} \quad \text{gives} \quad \frac{f(x_0 + \Delta x)}{x_0 + \Delta x} > \frac{f(x_0)}{x_0}. \]
Basic Question

- How does a dependent variable change when the independent variable changes given that we have started at some initial value of the independent variable?

- We will discover that an easy solution to this question comes from the use of the differential.
Solution using Difference quotient

We know the change in Y given a small change in X is
\[ \Delta y = f(x_1) - f(x_0) = f(x_0 + \Delta x) - f(x_0) \]
Multiplying righthand side by \( \Delta x/\Delta x \) gives
\[ \Delta y = \left( \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right) \Delta x \]
i.e. the difference quotient times the change in X.
This maybe difficult to calculate.
Define $dx$ as an arbitrary change in $x$ from its initial value $x_0$ and $dy$ as the resulting change in $y$ along the tangent line from the initial value of the function $y_0 = f(x_0)$. The differential of $y = f(x)$, $dy$, evaluated at $x$ is

$$dy = f'(x_0) \cdot dx.$$
Figure 6.9 Differential Approximation and Actual Change of a Function
**Economic Example**

Recall our tax revenue function, for a unitary elasticity demand function and supply of good perfectly elastic at $P$.

$$R = \frac{tC}{p+t}$$

If we let $p = 2$ and $C = 100$, our function becomes

$$R = \frac{100t}{2+t}$$

Our derivative was

$$\frac{dR}{dt} = \frac{200}{(2+t)^2}$$

Suppose $t$ goes from 2 to 3.

$$\Delta R = f(3) - f(2) = \frac{100 \cdot 3}{2 + 3} - \frac{100 \cdot 2}{2 + 2} = 60 - 50 = 10$$

Using our differential

$$dR = f'(2)dx = \frac{200}{(2+2)^2} \cdot 1 = 12.5$$
The extent to which the differential serves as a good approximation to the actual change of the value of a function for discrete changes in its argument depends upon the size of the change of the argument.

Table on next slide illustrates for our tax revenue function.
### Actual and Approximate Changes in Revenue

<table>
<thead>
<tr>
<th>t</th>
<th>R</th>
<th>ΔR</th>
<th>dR (estimated)</th>
<th>Difference %</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>0.0%</td>
</tr>
<tr>
<td>2.1</td>
<td>51.21951</td>
<td>1.219512</td>
<td>1.25</td>
<td>2.5%</td>
</tr>
<tr>
<td>2.5</td>
<td>55.55556</td>
<td>5.555556</td>
<td>6.25</td>
<td>12.5%</td>
</tr>
<tr>
<td>2.75</td>
<td>57.89474</td>
<td>7.894737</td>
<td>9.375</td>
<td>18.8%</td>
</tr>
<tr>
<td>3</td>
<td>60</td>
<td>10</td>
<td>12.5</td>
<td>25.0%</td>
</tr>
</tbody>
</table>
Quality of Differential approximation

The extent to which the differential serves as a good approximation to the actual change of the value of a function also depends upon the function itself. The differential approximation holds more closely for functions that are closer to being linear.

Consider \( y = 100 + 200x - \beta x^2 \)

\( dy = (200 - 2\beta x_0) dx \) and the actual change in \( y \) for change \( dx \) is

\( \Delta y = (200 - 2\beta x_0 - \beta dx) dx \)
Quality of Differential

Actual and Approximate Changes with Varying Nonlinearity ($x_0=10$ $dx=1$)

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\Delta y$ (actual)</th>
<th>$dy$ (estimated)</th>
<th>Difference (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200</td>
<td>200</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>158</td>
<td>160</td>
<td>1.3</td>
</tr>
<tr>
<td>4</td>
<td>116</td>
<td>120</td>
<td>3.5</td>
</tr>
<tr>
<td>6</td>
<td>74</td>
<td>80</td>
<td>8.1</td>
</tr>
<tr>
<td>8</td>
<td>32</td>
<td>40</td>
<td>25</td>
</tr>
</tbody>
</table>
Figure 6.11 Differential Approximation for Beta with Different Functions

(a) $\beta = 0$
- $dy = 200dx$
- $y = 100 + 200x$
- $(10, 2100)$
- $(11, 2300)$

(b) $\beta = 2$
- $dy = 160dx$
- $y = 100 + 200x - 2x^2$
- $(10, 1900)$
- $(11, 2060)$
- $(11, 2058)$

(c) $\beta = 8$
- $dy = 40dx$
- $y = 100 + 200x - 8x^2$
- $(10, 1300)$
- $(11, 1340)$
- $(11, 1332)$