Economics 2301

Lecture 17

Differential

Average and marginal functions

- The average function rises as long as the marginal function lies above it.
- The average function declines as long as the marginal function lies below it.

Average and marginal functions

If $f'(x_0) > \frac{f(x_0)}{x_0} \Rightarrow$ for a small change in x in the neighborhood of x_0, $\frac{f(x_0+\Delta x)}{x_0+\Delta x} > \frac{f(x_0)}{x_0}$.

Proof :

$$f'(x_0) > \frac{f(x_0)}{x_0} \Rightarrow \text{for small } \Delta x \ \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} > \frac{f(x_0)}{x_0} \ or$$

$$\frac{f(x_0+\Delta x)}{\Delta x} > \frac{f(x_0)}{x_0} + \frac{f(x_0)}{\Delta x} \quad or$$

$$\frac{f(x_0+\Delta x)}{\Delta x} > \frac{(x_0+\Delta x) f(x_0)}{x_0 \cdot \Delta x} \quad \text{Multipling each side by}$$

$$\frac{\Delta x}{x_0+\Delta x} \quad gives \quad \frac{f(x_0+\Delta x)}{x_0+\Delta x} > \frac{f(x_0)}{x_0}.$$

Basic Question

- How does a dependent variable change when the independent variable changes given that we have started at some initial value of the independent variable?
- We will discover that an easy solution to this question comes from the use of the **differential**.

Solution using Difference quotient

We know the change in Y given a small change in X is

$$\Delta y = f\left(x_1\right) - f\left(x_0\right) = f\left(x_0 + \Delta x\right) - f\left(x_0\right)$$

Multiplying righthand side by $\Delta x / \Delta x$ gives

$$\Delta y = \left(\frac{f\left(x_0 + \Delta x\right) - f\left(x_0\right)}{\Delta x}\right)\Delta x$$

i.e. the difference quotient times the change in X.

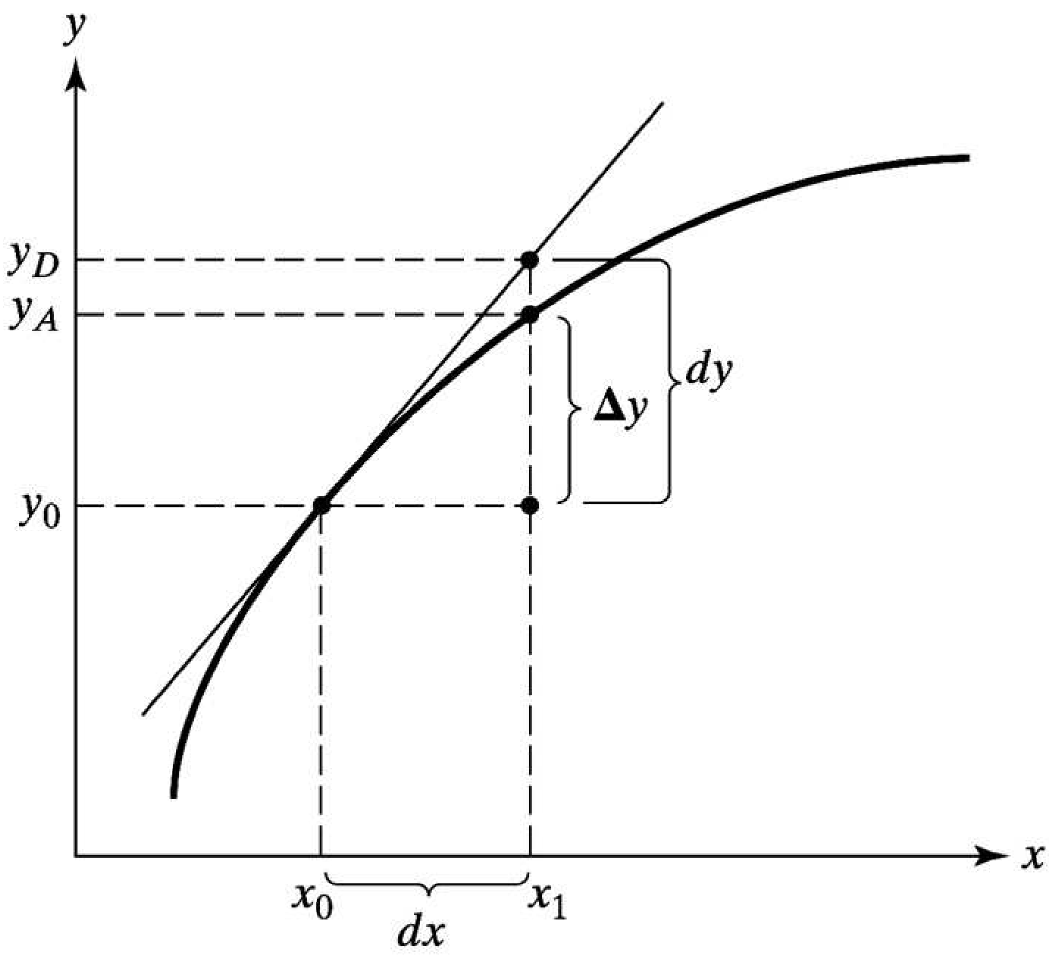
This maybe difficult to calculate.

The Differential

Define dx as an arbitrary change in x from its intial value x_0 and dy as the resulting change in y along the tangent line from the initial value of the function $y_0 = f(x_0)$. The differential of $y = f(x)$, dy, evaluated at x is

$$dy = f'(x_0) \cdot dx.$$

Figure 6.9 Differential Approximation and Actual Change of a Function



Economic Example

Recall our tax revenue function, for a unitary elasticity demand function and supply of good perfectly elastic at P.

$R = \frac{tC}{p+t}$ If we let p = 2 and C = 100, our function becomes

$$R = \frac{100t}{2+t}$$

Our derivative was $\frac{dR}{dt} = \frac{200}{(2+t)^2}$

Suppose t goes from 2 to 3.

$$\Delta R = f(3) - f(2) = \frac{100 \cdot 3}{2+3} - \frac{100 \cdot 2}{2+2} = 60 - 50 = 10$$

Using our differential

$$dR = f'(2)dx = \frac{200}{(2+2)^2} \cdot 1 = 12.5$$

Quality of Approximation

- The extent to which the differential serves as a good approximation to the actual change of the value of a function for discrete changes in its argument depends upon the size of the change of the argument.
- Table on next slide illustrates for our tax revenue function.

Actual an approximate changes in Revenue

t	R	ΔR	dR (estimated)	Difference %
2	50	0	0	0.0%
2.1	51.21951	1.219512	1.25	2.5%
2.5	55.55556	5.555556	6.25	12.5%
2.75	57.89474	7.894737	9.375	18.8%
3	60	10	12.5	25.0%

Quality of Differential approximation

The extent to which the differenti al serves as a good approximat ion to the actual change of the value of a function also depends upon the function itself.

The differenti al approximat ion holds more closely for functions that are closer to being linear.

Consider $y = 100 + 200x - \beta x^2$

$dy = (200 - 2\beta x_0)dx$ and the actual change in y for change dx is

$\Delta y = (200 - 2\beta x_0 - \beta dx)dx$

Quality of Differential

Actual and Approximate Changes with Varying Nonlinearity (x_0=10 dx=1)

β	Δy (actual)	dy (estimated)	Difference (%)
O	200	200	0
2	158	160	1.3
4	116	120	3.5
6	74	80	8.1
8	32	40	25

Figure 6.11 Differential Approximation for Beta with Different Functions