# Economics 2301

Lecture 16

Univariate Calculus

# At the Margin

- There is a direct correspondence between the mathematical concept of the derivative and the economic concept of considering decisions made “at the margin,” the point where the last unit is produced or consumed.
- This correspondence makes differentiation, which is the evaluation of a derivative, one of the core mathematical techniques in economics.

# Derivative Univariate Function

Recall that we difined the derivate of a univariate function

$y = f(x)$ evaluated at $x_0$ as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f\left(x_0 + \Delta x\right) - f\left(x_0\right)}{\Delta x}$$

# Sum-Difference Rule

For any two functions *f(x)* and *g(x)*

$$\frac{d\left(f\left(x\right)\pm g\left(x\right)\right)}{dx}=f'\left(x\right)\pm g'\left(x\right)$$

# Proof of Sum-Difference Rule

$$Let\ h(x) = f(x) \pm g(x)$$

$$h'(x) = \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(f(x_0+\Delta x) \pm g(x_0+\Delta x)\right) - \left(f(x_0) \pm g(x_0)\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{g(x_0+\Delta x) - g(x_0)}{\Delta x}$$

$$= f'(x) \pm g'(x)$$

# Example Sum-Difference Rule

$$Let\ h(x) = \left(3x^2 - 5x + 10\right) - \left(4 - 3x\right)$$

$$h'(x) = \left(6x - 5\right) - \left(-3\right) = 6x - 2$$

Note if we can rewrite h(x) as

$$h(x) = 3x^2 - 2x + 6$$

Using our rule for derivate of quadratic function, we get

$$h'(x) = 6x\text{-}2$$

# Scalar Rule

Let $g(x) = k \cdot f(x)$ where $k$ is a real number. Then

$$g'(x) = k \cdot f'(x)$$

Proof :

$$g'(x) = \lim_{\Delta x \to 0} \frac{k \cdot f(x_0 + \Delta x) - k \cdot f(x_0)}{\Delta x}$$

$$= k \cdot \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = k \cdot f'(x)$$

# Example Scalar Rule

$$let\ g(x) = 5 \cdot (10x^2 - 3x + 2)$$

$$g'(x) = 5 \cdot (20x - 3) = 100x - 15$$

Rewrite $g(x)$ as

$$g(x) = 50x^2 - 15x + 10 \quad then$$

$$g'(x) = 100x - 15$$

# Product Rule

$$For\ \ f(x) = g(x)\cdot h(x)$$

$$f'(x) = g'(x)\cdot h(x) + h'(x)\cdot g(x)$$

Proof :

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{g(x+\Delta x)h(x+\Delta x) - g(x)h(x)}{\Delta x}$$

Now add and substract $h(x)\cdot g(x+\Delta x)/\Delta x$

$$= \lim_{\Delta x\to 0} h(x)\cdot\left[\frac{g(x+\Delta x) - g(x)}{\Delta x}\right] + \lim_{\Delta x\to 0} g(x+\Delta x)\cdot\left[\frac{h(x+\Delta x) - h(x)}{\Delta x}\right]$$

$$= g'(x)\cdot h(x) + h'(x)\cdot g(x)$$

# Example Product Rule

$$f(x) = (2x+3)\cdot(5x-4)$$

$$f'(x) = 2\cdot(5x-4)+5\cdot(2x+3) = 10x-8+10x+15$$

$$= 20x+7$$

$$Note\ \ f(x) = 10x^2+7x-12$$

$$f'(x) = 20x+7$$

# Extension of Product Rule

$$let\ f(x) = g(x) \cdot h(x) \cdot n(x)$$

$$also\ let\ z(x) = h(x) \cdot n(x) \quad then$$

$$f'(x) = g(x) \cdot z'(x) + z(x) \cdot g'(x)$$

$$= g(x) \cdot [h'(x) \cdot n(x) + h(x) \cdot n'(x)] + [h(x) \cdot n(x)] \cdot g'(x)$$

$$= g'(x) \cdot h(x) \cdot n(x) + h'(x) \cdot g(x) \cdot n(x) + n'(x) \cdot g(x) \cdot h(x)$$

# Power Rule

$$\textit{Let} \quad f(x) = k \cdot x^n \quad \textit{then}$$

$$f'(x) = n \cdot k \cdot x^{n-1}$$

$$\text{Proof : let } g(x) = x^n$$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{C_0^n x^n \cdot (\Delta x)^0 + C_1^n x^{n-1} \cdot \Delta x + C_2^n x^{n-2} \cdot (\Delta x)^2 + \cdots + C_n^n x^0 (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[ C_1^n x^{n-1} + C_2^n x^{n-2} \cdot (\Delta x) + \cdots + C_n^n x^0 (\Delta x)^{n-1} \right]$$

$$= C_1^n x^{n-1} = n \cdot x^{n-1}$$

$$\therefore \quad f'(x) = k \cdot g'(x) = n \cdot k \cdot x^{n-1}$$

# Examples of power Rule

$$Let\ \ f(x) = k = k \cdot x^{0}$$

$$f'(x) = n \cdot k \cdot x^{n-1} = 0 \cdot k \cdot x^{-1} = 0$$

$$Let\ \ f(x) = \frac{k}{x^{n}} = k \cdot x^{-n} \quad then$$

$$f'(x) = -n \cdot k \cdot x^{-n-1} = \frac{-n \cdot k}{x^{n+1}}$$

# Examples of power Rule

$$Let\ \ f(x) = 6x^5 \quad then$$

$$f'(x) = 5 \cdot 6x^{5-1} = 30x^4$$

$$Let\ \ f(x) = 4 \cdot \sqrt[3]{x} = 4x^{1/3} \quad then$$

$$f'(x) = 1/3 \cdot 4 \cdot x^{1/3-1} = \frac{4}{3}x^{-2/3} = \frac{4}{3x^{2/3}}$$

# Exponential Function Rule

$$Let\ f(x) = e^{kx} \quad then\ f'(x) = k \cdot e^{kx}$$

proof : We will furnish once we have the chain rule.

# Critical limit

$$\lim_{h \to 0} \frac{\left(e^h - 1\right)}{h} = 1$$

Proof

the above limit means that for very very small values of

$e^h - 1$ is approximately $h$

$\Leftrightarrow \quad e^h$ is approximately $h + 1$

$\Leftrightarrow \quad e$ is approximately $(1 + h)^{1/h}$

but we know

$$e = \lim_{h \to 0} (1 + h)^{1/h}$$

# Derivate of $e^x$

$$Let\ f(x) = e^x \quad then$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x\left(e^{\Delta x} - 1\right)}{\Delta x}$$

$$= e^x \lim_{\Delta x \to 0} \frac{\left(e^{\Delta x} - 1\right)}{\Delta x} \text{ By the previous slide we get}$$

$$= e^x(1) = e^x$$

$$QED$$

# Growth Rate

Our continous growth model gave

$X(t) = X(0)e^{rt}$

$$now \quad \frac{dX(t)}{dt} = r \cdot X(0)e^{rt}$$

$$now \quad \frac{dX(t)}{dt} \Big/ X(t) = r$$

$r$ is sometimes called the intantaneous growth rate, since it is the growth rate over a very small unit of time. Recall proportial change

$$\frac{X_t - X_{t-1}}{X_{t-1}} = \frac{\Delta X}{X_{t-1}} = \frac{\Delta X/1}{X_{t-1}} = \frac{\Delta X/\Delta t}{X_{t-1}}$$

$now\ as\ \Delta t \to 0,\ X_{t-1} \to X_t$ hence the above ratio becomes

$$\frac{\Delta X/\Delta t}{X_{t-1}} \to \frac{\Delta X/\Delta t}{X_t} \xrightarrow[\Delta t \to 0]{} \frac{dX(t)}{dt} \Big/ X(t)$$