Economics 2301

Lecture 19

Univariate Calculus

Power Rule

$$Let \quad f(x) = k \cdot x^n \quad then$$

$$f'(x) = n \cdot k \cdot x^{n-1}$$

$$\text{Proof : let } g(x) = x^n$$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{C_0^n x^n \cdot (\Delta x)^0 + C_1^n x^{n-1} \cdot \Delta x + C_2^n x^{n-2} \cdot (\Delta x)^2 + \cdots + C_n^n x^0 (\Delta x)^n - x^n}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[C_1^n x^{n-1} + C_2^n x^{n-2} \cdot (\Delta x) + \cdots + C_n^n x^0 (\Delta x)^{n-1} \right]$$

$$= C_1^n x^{n-1} = n \cdot x^{n-1}$$

$$\therefore \quad f'(x) = k \cdot g'(x) = n \cdot k \cdot x^{n-1}$$

Examples of power Rule

$$Let\ \ f(x) = k = k \cdot x^{0}$$

$$f'(x) = n \cdot k \cdot x^{n-1} = 0 \cdot k \cdot x^{-1} = 0$$

$$Let\ \ f(x) = \frac{k}{x^{n}} = k \cdot x^{-n} \quad then$$

$$f'(x) = -n \cdot k \cdot x^{-n-1} = \frac{-n \cdot k}{x^{n+1}}$$

Examples of power Rule

$$Let \;\; f(x) = 6x^5 \quad then$$

$$f'(x) = 5 \cdot 6x^{5-1} = 30x^4$$

$$Let \;\; f(x) = 4 \cdot \sqrt[3]{x} = 4x^{1/3} \quad then$$

$$f'(x) = 1/3 \cdot 4 \cdot x^{1/3-1} = \frac{4}{3} x^{-2/3} = \frac{4}{3x^{2/3}}$$

Exponential Function Rule

$$Let\ \ f(x) = e^{kx} \quad then\ \ f'(x) = k \cdot e^{kx}$$

proof : We will furnish once we have the chain rule.

Critical limit

$$\lim_{h \to 0} \frac{\left(e^h - 1\right)}{h} = 1$$

Proof

the above limit means that for very very small values of

$e^h - 1$ is approximately h

$\Leftrightarrow \quad e^h$ is approximately $h + 1$

$\Leftrightarrow \quad e$ is approximately $(1 + h)^{1/h}$

but we know

$$e = \lim_{h \to 0} (1 + h)^{1/h}$$

Derivate of e^x

$$Let\ \ f(x) = e^x \quad then$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^x\left(e^{\Delta x} - 1\right)}{\Delta x}$$

$$= e^x \lim_{\Delta x \to 0} \frac{\left(e^{\Delta x} - 1\right)}{\Delta x} \text{ By the previous slide we get}$$

$$= e^x(1) = e^x$$

$$QED$$

Growth Rate

Our continous growth model gave

$$X(t) = X(0)e^{rt}$$

$$now \quad \frac{dX(t)}{dt} = r \cdot X(0)e^{rt}$$

$$now \quad \frac{dX(t)}{dt} \Big/ X(t) = r$$

r is sometimes called the intantaneous growth rate, since it is the growth rate over a very small unit of time. Recall proportial change

$$\frac{X_t - X_{t-1}}{X_{t-1}} = \frac{\Delta X}{X_{t-1}} = \frac{\Delta X / 1}{X_{t-1}} = \frac{\Delta X / \Delta t}{X_{t-1}}$$

$now\ as\ \Delta t \to 0,\ X_{t-1} \to X_t$ hence the above ratio becomes

$$\frac{\Delta X / \Delta t}{X_{t-1}} \to \frac{\Delta X / \Delta t}{X_t} \xrightarrow[\Delta t \to 0]{} \frac{dX(t)}{dt} \Big/ X(t)$$

Example Growth Rate

$$X(t) = 100e^{0.03t}$$

$$\frac{dX(t)}{dt} = 0.03 \cdot 100e^{0.03t} = 3e^{0.03t}$$

The instantaneous growth rate is

$$\frac{dX(t)}{dt} \Big/ X(t) = 3e^{0.03t} / 100e^{0.03t} = 0.03$$

$or\ 3\%$