Economics 2301

Lecture 20

Univariate Calculus
Chain Rule

- Many economic situations involve a chain of relationships that relate an ultimate outcome to its original cause.
- Money supply impact on output in Keynesian model.
- Chain rule shows how to evaluate a derivative when variable’s effect works through one function embedded in another.
Composite Functions

- The chain rule shows how to differentiate composite functions, that is, functions in which the arguments are themselves functions.

- Let \( y = g(u) \) and \( u = h(x) \), then the composite function \( f(x) \) is \( y = g(h(x)) = f(x) \).

- \( h(x) \) is the inside function and \( g(u) \) is the outside function.

- In general \( g(h(x)) \neq h(g(x)) \).
The derivative of the composite function
\[ y = f(x) = g(h(x)) \]
where
\[ u = h(x) \text{ and } g(h(x)) = g(u) \]
and both \( h(x) \) and \( g(u) \) are differentiable functions, is
\[
\frac{df(x)}{dx} = g'(h(x)) \cdot h'(x) \quad \text{or} \\
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]
Proof of Derivate of Exponential function

\[ \text{Let } y = e^{f(x)}, \text{ then } \frac{dy}{dx} = f'(x)e^{f(x)} \]

\text{proof :}

let \( u = f(x) \), then \( y = e^{f(x)} = e^u \)

by chain rule

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx} = e^u \cdot f'(x) = f'(x) \cdot e^{f(x)} \]

Using this result, we get when \( y = ke^{ax} \),

\[ \frac{dy}{dx} = a \cdot ke^{ax} \]
Example Chain Rule

\[ y = e^{(a+bx+cx^2)} \text{ then} \]

\[ \frac{dy}{dx} = \frac{d(a + bx + cx^2)}{dx} e^{(a+bx+cx^2)} \]

\[ = (b + 2cx) \cdot e^{(a+bx+cx^2)} \]
Let $y = (5 + 3x - 2x^2)^4 = u^4$ where $u = 5 + 3x - 2x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 4u^3 \cdot (3 - 4x) = 4 \cdot (3 - 4x) \cdot (5 + 3x - 2x^2)^3$$

$$= (12 - 16x) \cdot (5 + 3x - 2x^2)^3$$
General Power Rule

For the function \( f(x) = k[h(x)]^n \)
\[
f'(x) = k \cdot n \cdot [h(x)]^{n-1} \cdot h'(x)
\]
Example General Power Rule

\[ y = 0.5 \cdot (0.5x^{-5} - 0.5x^{-5})^{0.5} \]

Let \( h(x) = 0.5x^{-5} - 0.5x^{-5} \)

\[ y = 0.5 \cdot [h(x)]^{0.5} \]

\[ \frac{dy}{dx} = 0.5 \cdot 0.5 \cdot [h(x)]^{-5} \cdot h'(x) \]

\[ = 0.5 \cdot 0.5 \cdot [0.5x^{-5} - 0.5x^{-5}]^{-5} \cdot (0.5 \cdot 0.5x^{-5} - (-0.5) \cdot 0.5x^{-1.5}) \]

\[ = 0.25 \cdot [0.5x^{-5} - 0.5x^{-5}]^{-5} \cdot [0.25x^{-5} + 0.25x^{-1.5}] \]
The derivative of $f(x) = \frac{g(x)}{h(x)}$ is

$$f'(x) = \frac{g'(x)h(x) - h'(x)g(x)}{[h(x)]^2}$$

**proof:**

Let $w(x) = [h(x)]^{-1}$, then $f(x) = g(x) \cdot w(x)$ and

$$f'(x) = g'(x)w(x) + g(x)w'(x)$$

$$w'(x) = -1 \cdot [h(x)]^{-2} \cdot h'(x) = -h'(x)/[h(x)]^2$$

Substituting in we get

$$f'(x) = g'(x)w(x) + g(x)w'(x) = g'(x)/h(x) - g(x)h'(x)/[h(x)]^2$$

$$= \frac{g'(x)h(x) - h'(x)g(x)}{[h(x)]^2}$$

QED
Example of Quotient Rule

\[ y = e^{3x} / x^2 \]

\[
\frac{dy}{dx} = \frac{3e^{3x} \cdot x^2 - 2xe^{3x}}{(x^2)^2} = \frac{e^{3x} (3x^2 - 2x)}{x^4} = \frac{e^{3x} (3x - 2)}{x^3}
\]