Economics 2301

Lecture 20

Univariate Calculus

Chain Rule

- Many economic situations involve a chain of relationships that relate an ultimate outcome to its original cause.
- Money supply impact on output in Keynesian model.
- Chain rule shows how to evaluate a derivate when variable's effect works through one function embedded in another.

Composite Functions

- The chain rule shows how to differentiate **composite functions**, that is, functions in which the arguments are themselves functions.
- Let $y=g(u)$ and $u=h(x)$, then the composite function $f(x)$ is $y=g(h(x))=f(x)$.
- $h(x)$ is the **inside function** and $g(u)$ is the **outside function**.
- In general $g(h(x)) \neq h(g(x))$.

Chain Rule

The derivative of the composite function

$$y = f(x) = g(h(x))$$

where

$$u = h(x) \; and \; g(h(x)) = g(u)$$

and both *h(x)* and *g(u)* are differentiable functions, is

$$\frac{df(x)}{dx} = g'(h(x)) \cdot h'(x) \quad or$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Proof of Derivate of Exponential function

$$Let\ y = e^{f(x)}, then\ \frac{dy}{dx} = f'(x)e^{f(x)}$$

$$proof\ :$$

$$let\ u = f(x), then\ y = e^{f(x)} = e^{u}$$

$$by\ chain\ rule$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^{u} \cdot \frac{du}{dx} = e^{u} \cdot f'(x) = f'(x) \cdot e^{f(x)}$$

Using this result, we get when $y = ke^{ax}$,

$$\frac{dy}{dx} = a \cdot ke^{ax}$$

Example Chain Rule

$$y = e^{(a+bx+cx^2)} \ then$$

$$\frac{dy}{dx} = \frac{d(a+bx+cx^2)}{dx} e^{(a+bx+cx^2)}$$

$$= (b+2cx) \cdot e^{(a+bx+cx^2)}$$

2nd Example of Chain Rule

$$Let\ y = (5+3x-2x^2)^4 = u^4\ where\ u = 5+3x-2x^2$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 4u^3\cdot(3-4x) = 4\cdot(3-4x)\cdot(5+3x-2x^2)^3$$

$$= (12-16x)\cdot(5+3x-2x^2)^3$$

General Power Rule

For the function $f(x) = k\left[h(x)\right]^n$

$$f'(x) = k \cdot n \cdot \left[h(x)\right]^{n-1} \cdot h'(x)$$

Example General Power Rule

$$y = 0.5 \cdot (0.5x^{.5} - 0.5x^{-.5})^{0.5}$$

$$let\ h(x) = 0.5x^{.5} - 0.5x^{-.5}$$

$$y = 0.5 \cdot [h(x)]^{0.5}$$

$$\frac{dy}{dx} = 0.5 \cdot 0.5 \cdot [h(x)]^{-.5} \cdot h'(x)$$

$$= 0.5 \cdot 0.5 \cdot \left[0.5x^{.5} - 0.5x^{-.5}\right]^{-.5} \cdot \left(0.5 \cdot 0.5x^{-.5} - (-0.5) \cdot 0.5x^{-1.5}\right)$$

$$= 0.25 \cdot \left[0.5x^{.5} - 0.5x^{-.5}\right]^{-.5} \cdot \left[0.25x^{-.5} + 0.25x^{-1.5}\right]$$

Quotient Rule

The derivative of $f(x) = \frac{g(x)}{h(x)}$ *is*

$$f'(x) = \frac{g'(x)h(x) - h'(x)g(x)}{[h(x)]^2}$$

proof :

let $w(x) = [h(x)]^{-1}$, *then* $f(x) = g(x) \cdot w(x)$ *and*

$$f'(x) = g'(x)w(x) + g(x)w'(x)$$

$$w'(x) = -1 \cdot [h(x)]^{-2} \cdot h'(x) = -h'(x)/[h(x)]^2$$

Substituting in we get

$$f'(x) = g'(x)w(x) + g(x)w'(x) = g'(x)/h(x) - g(x)h'(x)/[h(x)]^2$$

$$= \frac{g'(x)h(x) - h'(x)g(x)}{[h(x)]^2}$$

QED

Example of Quotient Rule

$$y = e^{3x} / x^2$$

$$\frac{dy}{dx} = \frac{3e^{3x} \cdot x^2 - 2xe^{3x}}{\left(x^2\right)^2} = \frac{e^{3x}(3x^2 - 2x)}{x^4} = \frac{e^{3x}(3x - 2)}{x^3}$$