Economics 2301

Lecture 25

Risk Aversion
Risk Aversion

The curvature of a utility function is related to its implicit modeling of risk aversion.

Game: You can have $10 for sure, or get $0 if when you cut a deck of cards you get less than an ace, or receive $130 if you cut an ace.

\[
E(\text{return}) = \frac{12}{13} \cdot (0) + \frac{1}{13} \cdot (130) = 0 + \frac{130}{13} \approx 10
\]

This is a fair game.

If \( E(\text{Utility of the Game}) > \text{Utility(sure thing)} \), one is risk lover. If \( E(\text{Utility of the Game}) < \text{Utility (sure thing)} \) one is risk adverse.

\[
E(\text{Utility of Game}) = P(\text{Loss}) \cdot \text{Utility(loss)} + P(\text{win}) \cdot \text{Utility(win)}
\]
Figure 7.7 Utility Functions for Risk-Averse and Risk-Loving Individuals
Risk Adverse: Assume you have the following square root utility function: 

\[ U(m) = 5 \cdot \sqrt{m} \]

Utility sure thing: \( U(\$10) = 5 \cdot \sqrt{10} = 5 \cdot (3.1623) = 15.81 \)

Expected Utility of gamble: \( \frac{12}{13} \cdot U(0) + \frac{1}{13} \cdot U(130) \)

\[
= \frac{12}{13} \cdot 5 \sqrt{0} + \frac{1}{13} \cdot 5 \sqrt{130} = 0 + \frac{5 \cdot 11.402}{13} = 4.39 < 15.81
\]

Person will select the sure thing because it maximizes their expected utility and they are risk adverse.
Proof of Risk Adverse

\[
U(m) = 5 \cdot \sqrt{m}
\]

\[
U'(m) = \frac{5}{2} \cdot (m)^{-\frac{1}{2}}
\]

\[
U''(m) = -\frac{5}{4} \cdot (m)^{-\frac{3}{2}} < 0
\]

The utility function is concave and hence risk adverse.
Example-Risk Lover

For our risk lover assume the individual has the following utility function: \( U(m) = 5 \cdot m^{1.2} \)

Utility sure thing: \( U(10) = 5 \cdot (10)^{1.2} = 79.24 \)

Utility of the gamble: \( E(U(m)) = \frac{12}{13} \cdot 5 \cdot (0)^{1.2} + \frac{1}{13} \cdot 5 \cdot (30)^{1.2} \)

\[ = 0 + \frac{5 \cdot 344.1375}{13} = 132.36 > 79.24 \]

\[ \therefore \text{Our risk lover selected the gamble as it maximizes his expected utility.} \]
Proof Risk Lover

\[ U(m) = 5 \cdot (m)^{1.2} \]
\[ U'(m) = 6 \cdot (m)^2 \]
\[ U''(m) = 1.2 \cdot (m)^{-0.8} > 0 \]

Function is convex and the person is a risk lover.
Person who is both risk-lover and risk-adverse
Plot of Marginal Utility and Second Derivative
Proof of Properties of both risk-lover and risk-adverse

\[ U(C) = 0.24 \cdot C^3 - 0.002 \cdot C^4 \]
\[ U'(C) = 0.72 \cdot C^2 - 0.008C^3 \]
\[ Roots = 0.90 \]
\[ U''(C) = 1.44 \cdot C - 0.024C^2 \]
\[ Roots = 0.60 \]

Our original function maximum at 90 and inflection point at 0 and 60.
Function is convex from 0 to 60 and concave beyond 60.
Linearization of Functions and Taylor Series

- It is often useful to be able to express a function in linear form.
- Helpful if we want to use matrix algebra or simple econometrics.
- Calculus can give us a linear approximation to a function.
Figure 7.8  Taylor Series Approximations
We have the function \( y = f(x) \) that we wish to approximate by a linear function at the point \( a \). The simpliest approximation is \( g(x) = f(a) \). Not good when we get any distance from point. We prefer a linear function of the form

\[
h(x) = f(a) + b \cdot (x-a).
\]

Question? How do we select "\( b \)".

The differential is the equation for the tangent line to the function at the point \( x = a \). Therefore the best linear approximation is

\[
h(x) = f(a) + f'(a)(x-a).
\]
A better approximation allows for some curvature. Quadratic approximation would be
\[ j(x) = f(a) + f'(a)(x-a) + c(x-a)^2. \]
Want \( c \) related to the second derivative of the function at \( x = a \).

\[
\begin{align*}
J''(a) &= 2c = f''(a) \implies c = \frac{f''(a)}{2}\\
\therefore \quad j(x) &= f(a) + f'(a)(x-a) + \frac{1}{2} f''(a) \cdot (x-a)^2.
\end{align*}
\]
The $n^{th}$ degree approximation to our function at $x = a$ is

$$m(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!} (x - a)^n$$

Where

$f^{(n)}(a) =$ nth derivative of $f(x)$ evaluated at $x = a.$

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$

$0! = 1$

$1! = 1$
Example

Consider $y = e^{x^2}$ at the point $x = 2$.

Linear: $h(x) = e^4 + 4e^4 \cdot (x - 2) = -7e^4 + 4e^4 \cdot x$

Quadratic: $j(x) = e^4 + 4e^4 \cdot (x - 2) + 9e^4 \cdot (x - 2)^2$
Graph of our approximations
## Accuracy of our approximation

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<th>x</th>
<th>f(x)</th>
<th>h(x)</th>
<th>j(x)</th>
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<th>Quadratic % Error</th>
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