Economics 2301

Lecture 26

Ceteris Paribus

- Economic analysis often proceeds by considering the consequences of a certain event, *ceteris paribus*.
- The advantage of this approach is that it identifies the exclusive impact of the variable under study rather than confounding its effect with the effect of other variables.

Ceteris Paribus and Multivariate Functions

Suppose you have the following demand function :

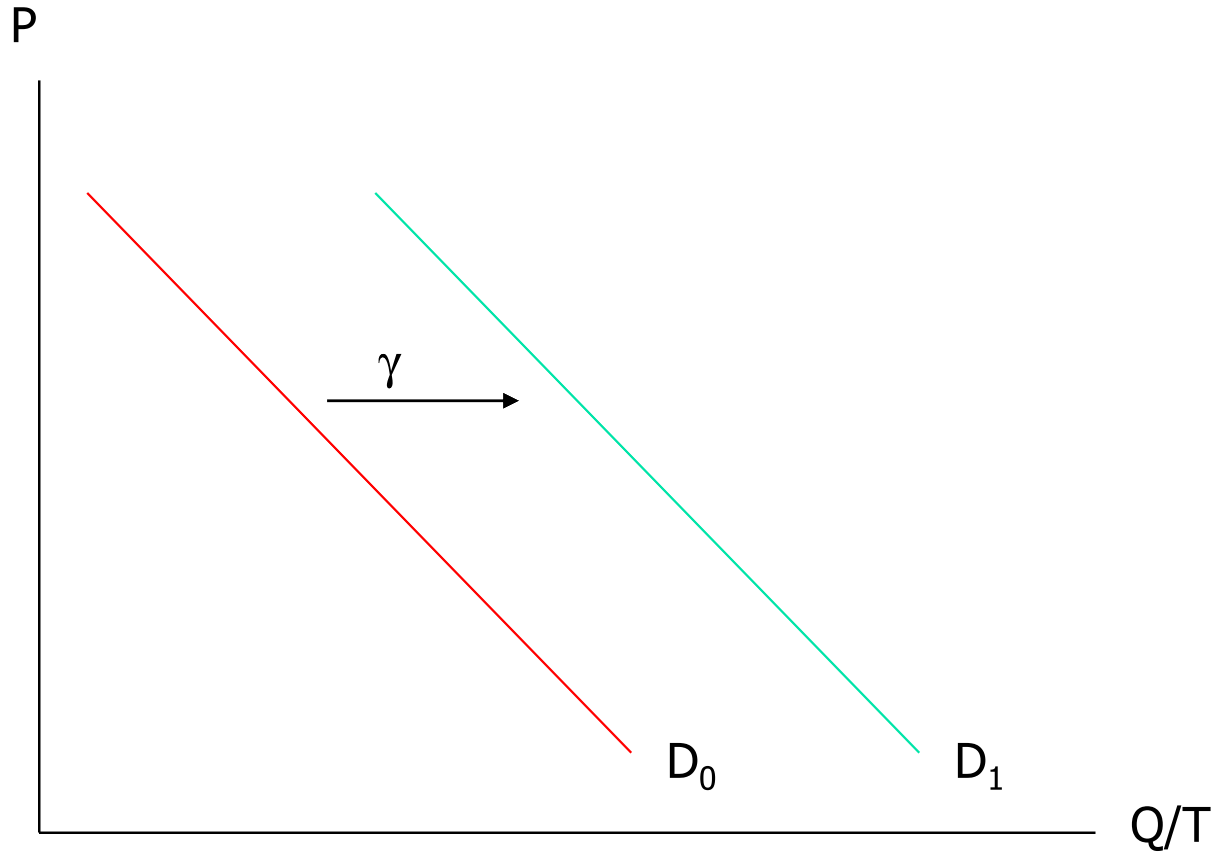
$\mathrm{Q} = \alpha - \beta\mathrm{P} + \gamma\mathrm{I} + \delta\mathrm{P_S}$ *where* Q = Quantity, P = own price, I = income and $\mathrm{P_S}$ = price of substitute good.

The effect of a 1 unit increase in income holding price of the good and price of substitute constant is

$$\frac{\Delta \mathrm{Q}}{\Delta \mathrm{I}} = \frac{Q_{I+1} - Q_I}{1} = \frac{(\alpha - \beta\mathrm{P} + \gamma(\mathrm{I}+1) + \delta\mathrm{P_S}) - (\alpha - \beta\mathrm{P} + \gamma\mathrm{I} + \delta\mathrm{P_S})}{1}$$

$$= \gamma$$

Graph of income effect



Demand Example Continued

Now let the demand equation be

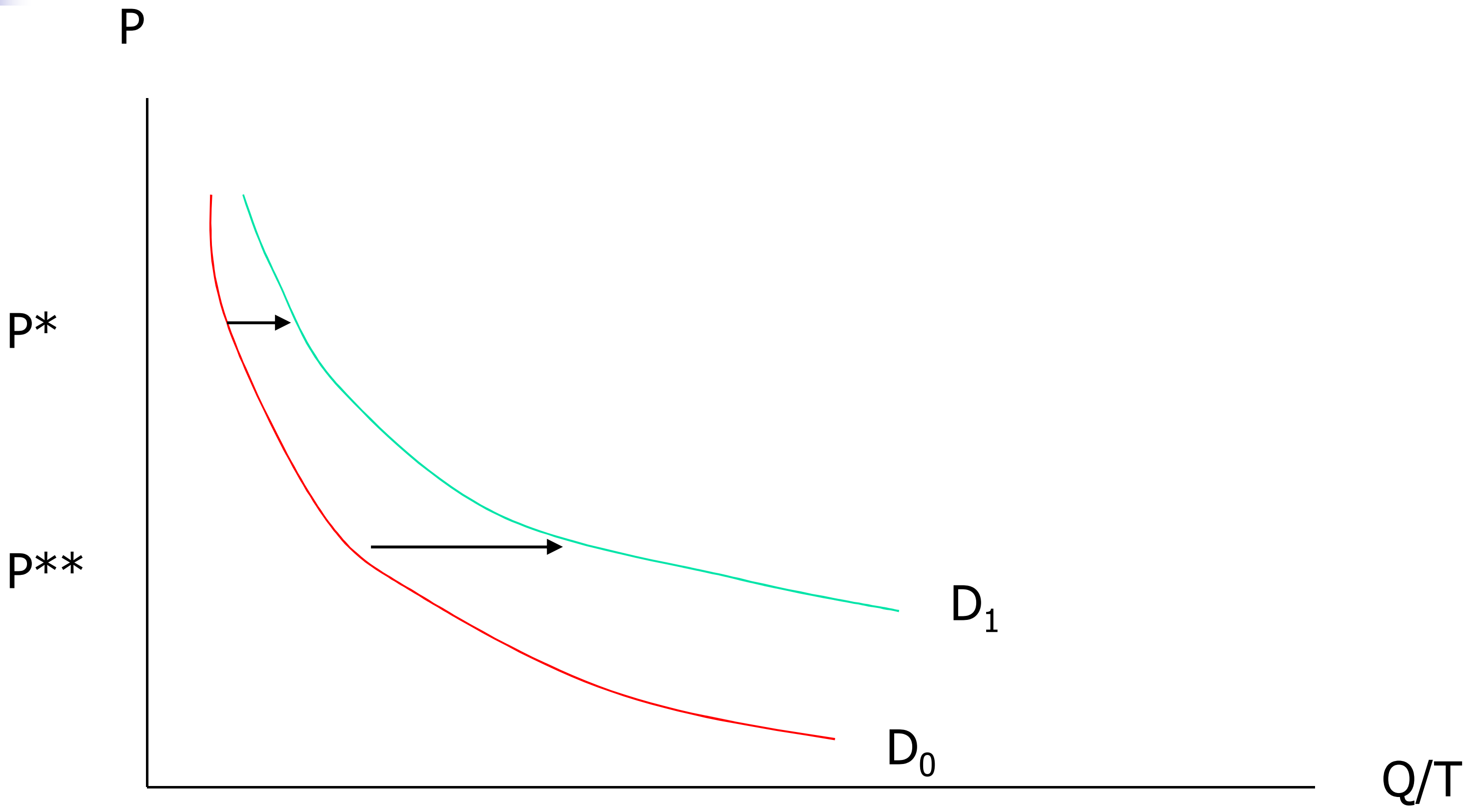
$$Q = {100I}\Big/{(5P - 2P_S)}$$

The difference in the quantity demanded given a change in income is now

$$\frac{\Delta Q}{\Delta I} = \frac{100(I+1)/(5P-2P_S) - 100I/(5P-2P_S)}{I+1-I} = \frac{100}{(5P-2P_S)}$$

The gain from an increase income decreases as the price of the product rises and rises as its substitute's price rises.

Effect of income change on Demand for model 2



Partial Derivatives

A partial derivative of a multivariate function with respect to any one of its arguments represents the rate of change of the value of that function due to a very small change in that argument while all the other variables that are also arguments of this function are held constant. The difference quotient of the previous example introduces the *ceteris paribus* characteristic of a partial derivative.

Partial Derivative

The partial derivative of the function $y = f(x_1, x_2, \ldots, x_n)$ with respect to its argument x_j, written as $\partial y / \partial x_j$ is

$$\frac{\partial y}{\partial x_j} = \lim_{\Delta x_j \to 0} \frac{f\left(x_1, \cdots, x_j + \Delta x_j, \cdots, x_n\right) - f\left(x_1, \cdots, x_j, \cdots, x_n\right)}{\Delta x_j}$$

provided the limit exists.

We may also denote the partial derivative of y with respect to x_j as $f_j(x_1, x_2, \ldots, x_n)$ or f_j.

Partial Derivative

- The partial derivative of a multivariate function with respect to one of its arguments is found by applying the rules for univariate differentiation and treating all the other arguments of the function as constant.

Examples

$$Let\ y = \frac{3x_1^2}{\sqrt{x_2}}$$

$$\frac{\partial y}{\partial x_1} = \frac{6x_1}{\sqrt{x_2}} \quad and \quad \frac{\partial y}{\partial x_2} = -\frac{3x_1^2}{2(x_2)^{3/2}}$$

$$Let\ y = e^{x_1 x_2}$$

$$\frac{\partial y}{\partial x_1} = x_2 e^{x_1 x_2} \quad and \quad \frac{\partial y}{\partial x_2} = x_1 e^{x_1 x_2}$$

Economic Example

Suppose you have the following utility function :

$$U = U\left(x_1, x_2\right) = 200\ln(x_1) + 200\ln(x_2) - x_1 x_2$$

Marginal Utility of good x_1

$$U_1 = \frac{\partial U}{\partial x_1} = \frac{200}{x_1} - x_2$$

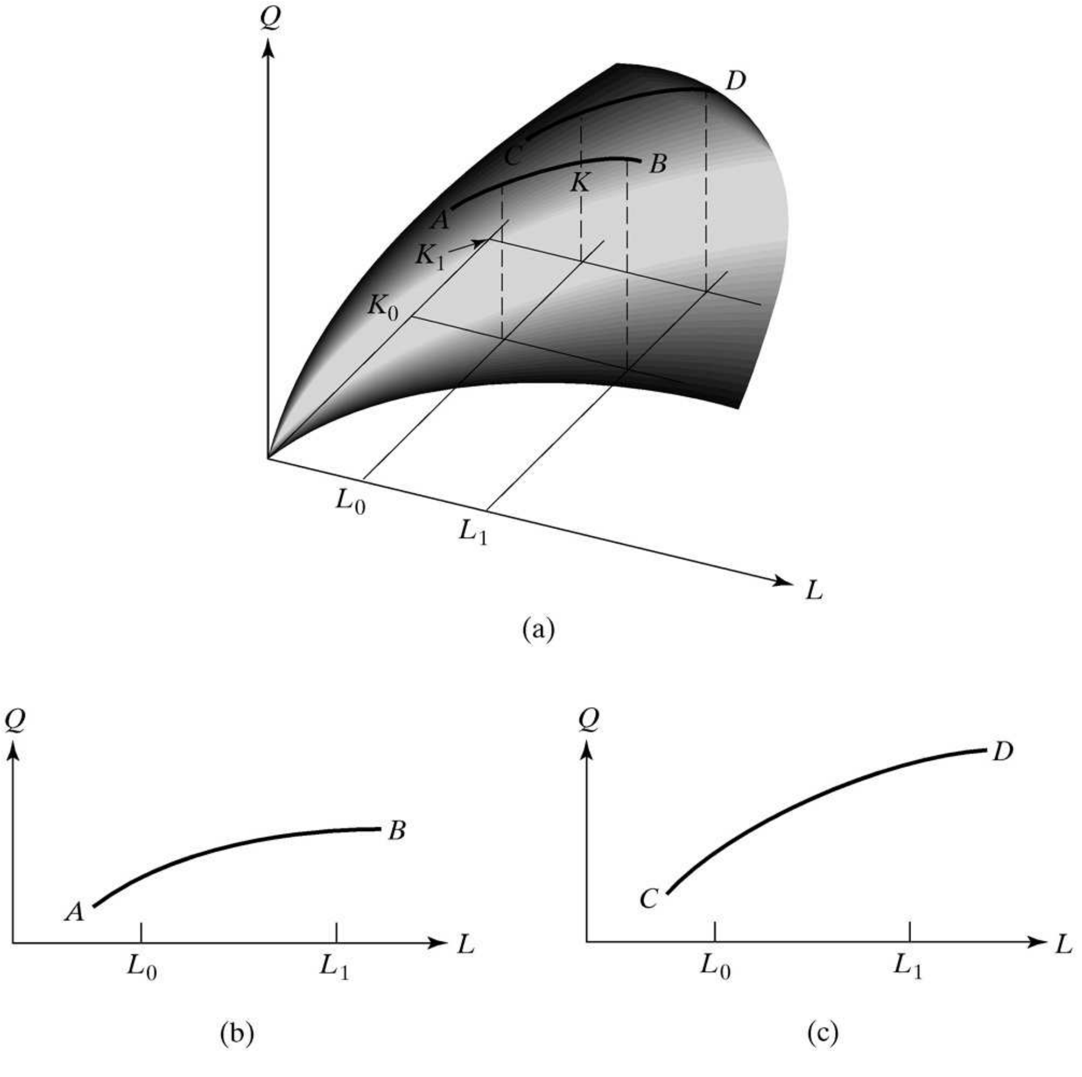
Marginal Utility of good x_2

$$U_2 = \frac{\partial U}{\partial x_2} = \frac{200}{x_2} - x_1$$

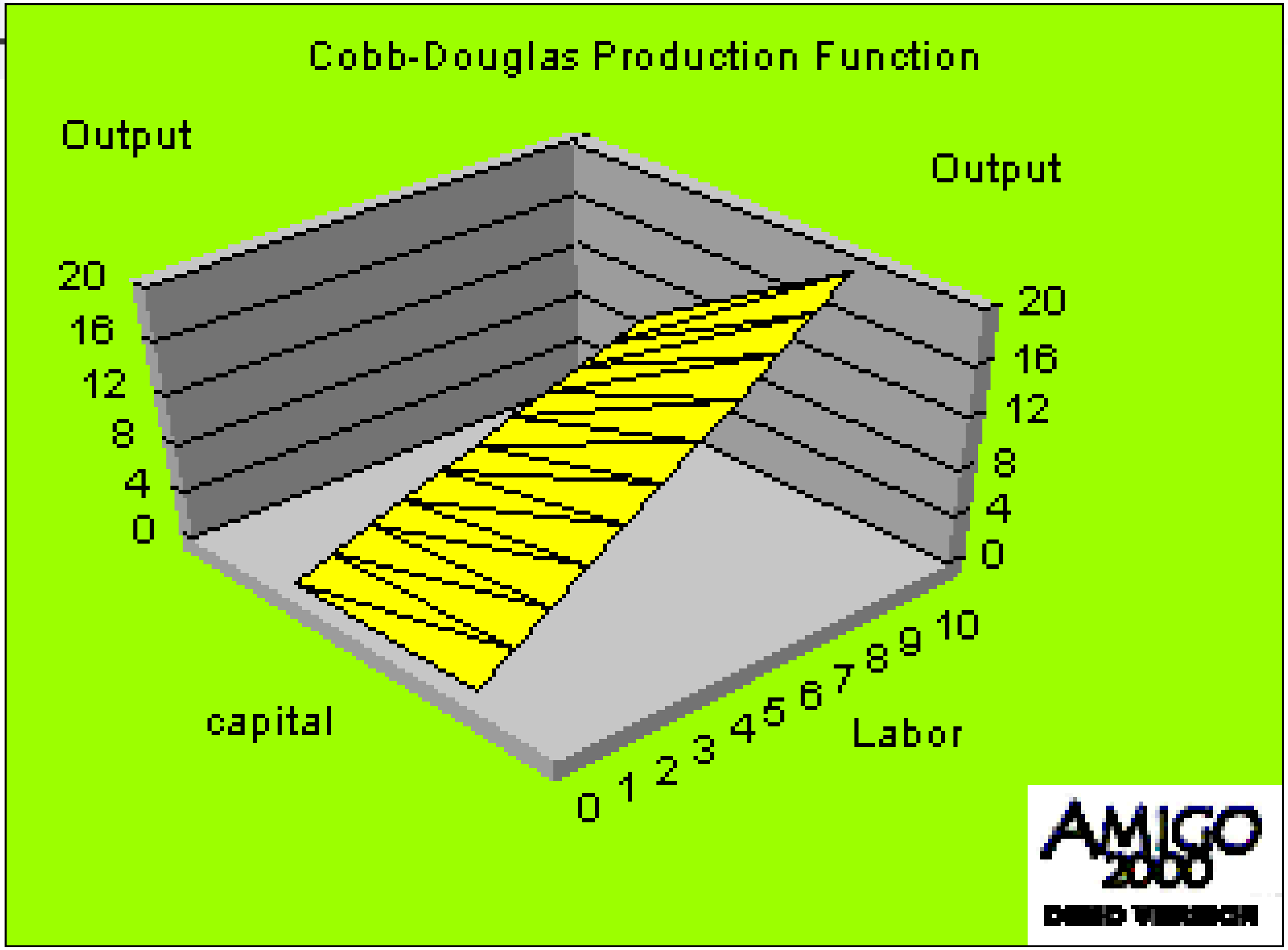
Geometric Interpretation of Partial Derivative

- As a special type of derivative, a partial derivative is a measure of the instantaneous rate of change of some variable, and in that capacity it again has a geometric counterpart in the slope of a particular curve.

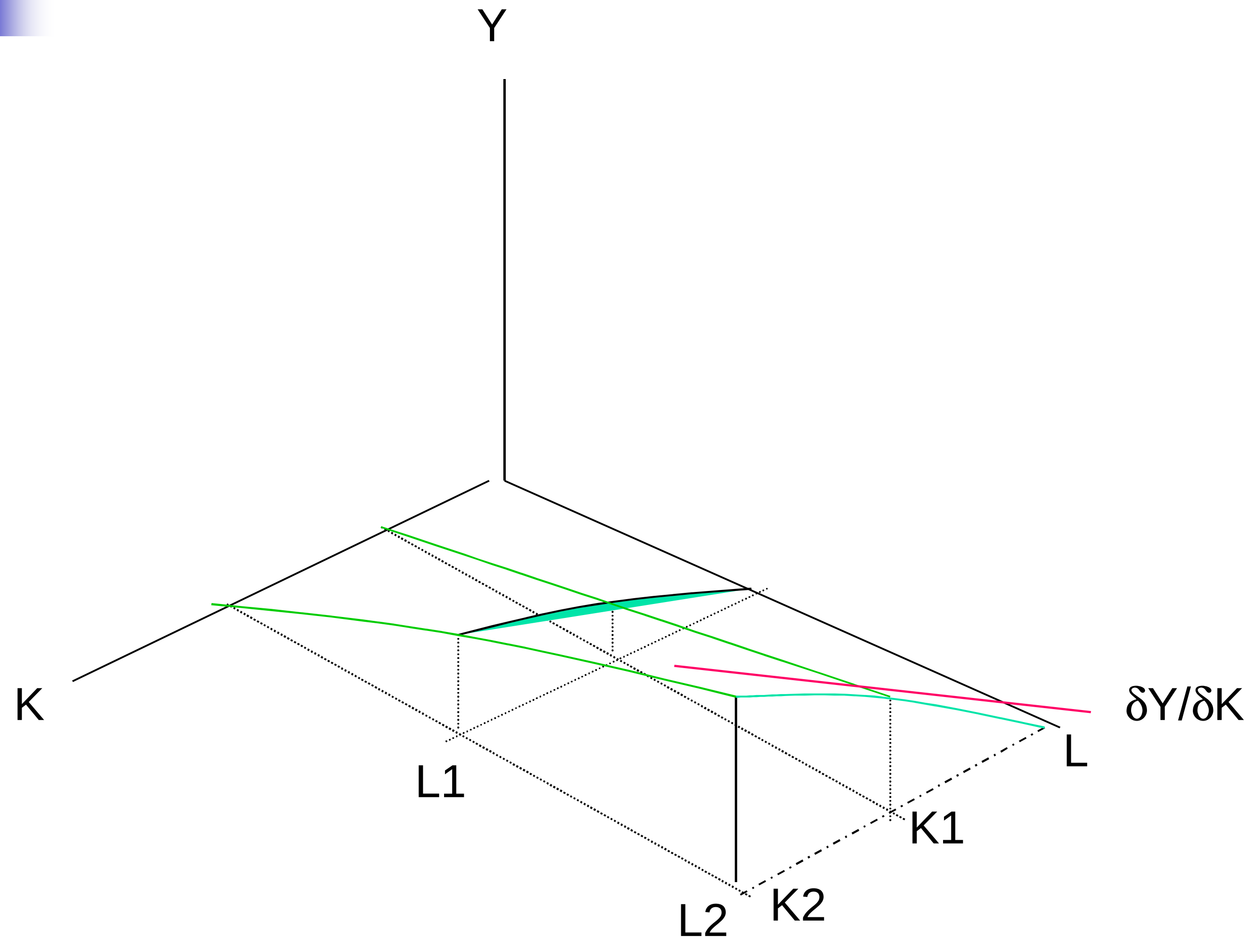
A Production Function and Its "Slices"



3 D graph of Production Function



Graph of Partial Derivative of Y with respect to K



Second Partial Derivative & Cross Partial Derivatives

- A second partial derivative is a measure of how a partial derivative with respect to one argument of a multivariate function changes with a very small change in that argument.
- A cross partial derivative is a measure of how a partial derivative taken with respect to one of the arguments of the multivariate function varies with a very small change in another argument of that function.

Partial Derivatives

We have the function $y = f(x_1,x_2)$

The two partial derivates are $f_1(x_1,x_2)$ *and* $f_2(x_1,x_2)$ *or*

$$\frac{\partial y}{\partial x_1} \text{ and } \frac{\partial y}{\partial x_2}$$

Second partial derivatives

$$f_{11}(x_1,x_2) = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_1}\right) = \frac{\partial^2 y}{\partial x_1^2}$$

$$f_{22}(x_1,x_2) = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_2^2}$$

Cross Partial Derivative

The cross partial derivative representing the partial derivative of $f_1(x_1,x_2)$ taken with respect to x_2 is denote as

$$f_{21}(x_1,x_2) = \frac{\partial}{\partial x_2}\left(\frac{\partial y}{\partial x_1}\right) = \frac{\partial^2 y}{\partial x_2 \partial x_1}$$

and the cross partial derivate of $f_2(x_1,x_2)$ taken with respect to x_1 is denoted as

$$f_{12}(x_1,x_2) = \frac{\partial}{\partial x_1}\left(\frac{\partial y}{\partial x_2}\right) = \frac{\partial^2 y}{\partial x_1 \partial x_2}$$

Example Utility Function

$$U = U(X,Y) = \beta X^{\alpha} Y^{1-\alpha} \quad 0 < \alpha < 1 \; and \;\; \beta > 0$$

Marginal Utilities

$$U_X = \frac{\partial U}{\partial X} = \alpha\beta X^{\alpha-1} Y^{1-\alpha} = \alpha\beta \left(\frac{Y}{X} \right)^{1-\alpha} > 0$$

$$U_Y = \frac{\partial U}{\partial Y} = (1-\alpha)\beta X^{\alpha} Y^{-\alpha} = (1-\alpha)\beta \left(\frac{X}{Y} \right)^{\alpha} > 0$$

Diminish Marginal Utility or second partial derivatives

$$U_{XX} = \frac{\partial^2 U}{\partial X^2} = (\alpha - 1)\alpha\beta X^{\alpha-2} Y^{1-\alpha} < 0$$

$$U_{YY} = \frac{\partial^2 U}{\partial Y^2} = -\alpha(1-\alpha)\beta X^{\alpha} Y^{-\alpha-1} < 0$$

Example Cross Partial

Marginal Utilities

$$U_X = \frac{\partial U}{\partial X} = \alpha\beta X^{\alpha-1} Y^{1-\alpha} = \alpha\beta\left(\frac{Y}{X}\right)^{1-\alpha} > 0$$

$$U_Y = \frac{\partial U}{\partial Y} = (1-\alpha)\beta X^{\alpha} Y^{-\alpha} = (1-\alpha)\beta\left(\frac{X}{Y}\right)^{\alpha} > 0$$

The cross partials are

$$U_{YX} = \frac{\partial}{\partial Y}\left(\frac{\partial U}{\partial X}\right) = \frac{\partial^2 U}{\partial Y \partial X} = (1-\alpha)\alpha\beta X^{\alpha-1} Y^{-\alpha} > 0$$

$$U_{XY} = \frac{\partial}{\partial X}\left(\frac{\partial U}{\partial Y}\right) = \frac{\partial^2 U}{\partial X \partial Y} = \alpha(1-\alpha)\beta X^{\alpha-1} Y^{-\alpha} > 0$$

These goods are compliment s in consumptio n. More of one increases the marginal utility of the other.