Economics 2301

Lecture 27

Multivariate Differential Calculus
Young’s Theorem

If all the partial derivatives of the function 
\( f(x_1, x_2, \cdots, x_n) \) exist and are themselves differentiable with
continuous derivates, then

\[
\frac{\partial}{\partial x_i} \left( \frac{\partial f(x_1, x_2, \cdots, x_n)}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left( \frac{\partial f(x_1, x_2, \cdots, x_n)}{\partial x_i} \right) \quad \text{or}
\]

\( f_{ij}(x_1, x_2, \cdots, x_n) = f_{ji}(x_1, x_2, \cdots, x_n) \)

for any \( i \) and \( j \) from 1 to \( n \).
Example Young’s Theorem

Let \( y = 3x^2 - 2xz - 4z^2 \)

First partial derivative is
\[
\frac{\partial y}{\partial x} = 6x - 2z \quad \text{and} \quad \frac{\partial y}{\partial z} = -2x - 8z
\]

Cross partial derivative is
\[
\frac{\partial}{\partial z} \left( \frac{\partial y}{\partial x} \right) = \frac{\partial^2 y}{\partial z \partial x} = -2 \quad \text{and} \quad \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial z} \right) = \frac{\partial^2 y}{\partial x \partial z} = -2
\]

The two cross partial derivatives are equal in accordance to Young’s Theorem.
From our previous lecture, we had the utility function
\[ U = \beta X^\alpha Y^{1-\alpha} \]
The marginal utilities were
\[ U_x = \alpha \beta X^{\alpha-1} Y^{1-\alpha} \quad \text{and} \quad U_y = (1-\alpha) \beta X^\alpha Y^{-\alpha} \]
The cross partial derivatives are
\[ U_{yx} = (1-\alpha) \alpha \beta X^{\alpha-1} Y^{-\alpha} \quad \text{and} \quad U_{xy} = \alpha (1-\alpha) \beta X^{\alpha-1} Y^{-\alpha} \]
The cross partial derivatives are equal and Young's Theorem holds.
Multivariate Chain Rule (I)

If the arguments of the differentiable function $y = f(x_1, x_2, \cdots, x_n)$ are themselves differentiable functions of the variable $t$ such that $x_1 = g^1(t), x_2 = g^2(t), \cdots, x_n = g^n(t)$, then

$$\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt} + \cdots + f_n \frac{dx_n}{dt}$$

where $f_i = \frac{\partial y}{\partial x_i}$. 
Economic Example

Suppose the production function for a society was

\[ Y = \alpha K^\beta L^{1-\beta} \]

and capital and labor grow at a constant rate through time, \( t \).

\[ K(t) = \delta e^{r_K t} \quad \text{and} \quad L(t) = \gamma e^{r_L t} \quad \text{then} \]

\[
\frac{dY}{dt} = \beta \alpha K^{\beta-1} L^{-\beta} \cdot r_K \delta e^{r_K t} + (1 - \beta) \alpha K^\beta L^{-\beta} \cdot r_L \delta e^{r_L t}
\]

\[ = Y \left( \beta r_K + (1 - \beta) r_L \right) \quad \text{and the rate of growth of output is} \]

\[
\frac{dY}{dt} \cdot \frac{1}{Y} = \beta r_K + (1 - \beta) r_L
\]
Alternative route to solution

By substituting the growth functions for labor and capital into our production function, we can get the growth function for output

\[ Y(t) = \alpha K^\beta L^{1-\beta} = \alpha \left( \delta e^{r_K t} \right) ^\beta \left( \gamma e^{r_L t} \right) ^{1-\beta} = \alpha \delta^\beta \gamma^{1-\beta} e^{(\beta r_K + (1 - \beta) r_L) t} \]

From univariate theory, we know that the growth rate of output is

\[ \frac{dY(t)}{dt} \cdot \frac{1}{Y(t)} = \beta r_K + (1 - \beta) r_L \]
If the arguments of the differentiable function
\[ y = f(x_1, x_2, \ldots, x_n) \] are themselves differentiable functions
of the variables, \( t_1, t_2, \ldots, t_m \) such that
\[ x_i = g^i(t_1, t_2, \ldots, t_m) \quad \text{for} \quad i = 1, 2, \ldots, n \quad \text{then} \]
\[ \frac{\partial y}{\partial t_i} = f_1 \frac{\partial g^1}{\partial t_i} + f_2 \frac{\partial g^2}{\partial t_i} + \cdots + f_n \frac{\partial g^n}{\partial t_i} \]
\[ = \sum_{j=1}^{n} f_j g^j_i \]
where \( f_j = \frac{\partial y}{\partial x_j} \quad \text{and} \quad g^j_i = \frac{\partial g^j}{\partial t_i} \)
Early Retirement

Suppose the rate of early retirements (prior to age of full social security benefits) is a function of the level of unemployment and the rate of inflation $R = R(U, \pi)$ and that unemployment and inflation are themselves functions of tax policy and monetary policy, $U = U(T, M)$ and $\pi = \pi(T, M)$ then

$$\frac{\partial R}{\partial T} = R_u^+ \cdot U_T^- + R_{\pi}^- \cdot \pi_T^- > 0$$

$$\frac{\partial R}{\partial M} = R_u^+ \cdot U_M^- + R_{\pi}^+ \cdot \pi_M^- < 0$$

Both effects are determinate.