Economics 2301

Lecture 27

Multivariate Differential Calculus

Young's Theorem

If all the partial derivatives of the function

$f(x_1, x_2, \cdots, x_n)$ exist and are themselves differentiable with

continuous derivates, then

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f(x_1, x_2, \cdots, x_n)}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f(x_1, x_2, \cdots, x_n)}{\partial x_i}\right) \; or$$

$$f_{ij}(x_1, x_2, \cdots, x_n) = f_{ji}(x_1, x_2, \cdots, x_n)$$

for any i and j from 1 to n.

Example Young's Theorem

$$Let \;\; y = 3x^2 - 2xz - 4z^2$$

$$First \;\; partial \;\; derivative \; s$$

$$\frac{\partial y}{\partial x} = 6x - 2z \quad and \quad \frac{\partial y}{\partial z} = -2x - 8z$$

$$Cross \;\; partial \;\; derivative \; s$$

$$\frac{\partial}{\partial z}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial z \partial x} = -2 \quad and \quad \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial z}\right) = \frac{\partial^2 y}{\partial x \partial z} = -2$$

The two cross partial derivative s are equal in accordance to Young' s Theorem.

Example 2 – Utility Function

From our previous lecture, we had the utility function

$$U = \beta X^{\alpha} Y^{1-\alpha}$$

The marginal utilities were

$$U_X = \alpha\beta X^{\alpha-1} Y^{1-\alpha} \quad and \quad U_Y = (1-\alpha)\beta X^{\alpha} Y^{-\alpha}$$

The crosss partial derivatives are

$$U_{YX} = (1-\alpha)\alpha\beta X^{\alpha-1} Y^{-\alpha} \quad and \quad U_{XY} = \alpha(1-\alpha)\beta X^{\alpha-1} Y^{-\alpha}$$

The cross partial derivatives are equal and Young's Theorem holds.

Multivariate Chain Rule (I)

If the arguments of the differentiable function $y = f(x_1, x_2, \cdots, x_n)$ are themselves differntiable functions of the the variable t such that

$$x_1 = g^1(t),\ x_2 = g^2(t), \cdots, x_n = g^n(t)$$

where $g^i(t)$ is the ith univariate function, then

$$\frac{dy}{dt} = f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt} + \cdots + f_n \frac{dx_n}{dt}$$

$$where\ \ f_i = \frac{\partial y}{\partial x_i}.$$

Economic Example

Suppose the production function for a society was

$Y = \alpha K^{\beta} L^{1-\beta}$ and capital and labor grow at a constant rate through time, t.

$K(t) = \delta e^{r_K t}$ *and* $L(t) = \gamma e^{r_L t}$ *then*

$$\frac{dY}{dt} = \beta\alpha K^{\beta-1} L^{1-\beta} \cdot r_K \delta e^{r_K t} + (1-\beta)\alpha K^{\beta} L^{-\beta} \cdot r_L \delta e^{r_L t}$$

$= Y\left(\beta r_K + (1-\beta) r_L\right)$ and the rate of growth of output is

$$\frac{dY}{dt} \cdot \frac{1}{Y} = \beta r_K + (1-\beta) r_L$$

Alternative route to solution

By substituting the growth functions for labor and capital into our production function, we can get the growth function for output

$$Y(t) = \alpha K^{\beta} L^{1-\beta} = \alpha\left(\delta e^{r_K t}\right)^{\beta}\left(\gamma e^{r_L t}\right)^{1-\beta} = \alpha\delta^{\beta}\gamma^{1-\beta} e^{(\beta r_K + (1-\beta) r_L)t}$$

From univariate theory, we know that the growth rate of output is

$$\frac{dY(t)}{dt} \cdot \frac{1}{Y(t)} = \beta r_K + (1-\beta) r_L$$

Multivariate Chain Rule (II)

If the arguments of the differentiable function

$y = f\left(x_1, x_2, \cdots, x_n\right)$ are themselves differentiable functions

of the vairables, $t_1, t_2, \cdots, t_m$ such that

$x_i = g^i\left(t_1, t_2, \cdots, t_m\right)$ *for* $i = 1, 2, \cdots, n$ *then*

$$\frac{\partial y}{\partial t_i} = f_1 \frac{\partial g^1}{\partial t_i} + f_2 \frac{\partial g^2}{\partial t_i} + \cdots + f_n \frac{\partial g^n}{\partial t_i}$$

$$= \sum_{j=1}^{n} f_j g_i^j$$

where $f_j = \dfrac{\partial y}{\partial x_j}$ *and* $g_i^j = \dfrac{\partial g^j}{\partial t_i}$

Early Retirement

Suppose the rate of early retirements (prior to age of full social security benefits) is a function of the level of unemployment and

the rate of inflation $R = R(\overset{+}{U}, \overset{-}{\pi})$ and that unemployment and inflation are themselves functions of tax policy and

monetary policy, $U = U\left(\overset{+}{T}, \overset{-}{M}\right)$ *and* $\pi = \pi\left(\overset{-}{T}, \overset{+}{M}\right)$ *then*

$$\frac{\partial R}{\partial T} = \overset{+}{R_U} \cdot \overset{+}{U_T} + \overset{-}{R_\pi} \cdot \overset{-}{\pi_T} > 0$$

$$\frac{\partial R}{\partial M} = \overset{+}{R_U} \cdot \overset{-}{U_M} + \overset{-}{R_\pi} \cdot \overset{+}{\pi_M} < 0$$

Both effects are determinate.