Economics 2301

Lecture 28

Multivariate Calculus

Homogeneous Function

A function $y = f(x_1, x_2, \cdots, x_n)$ is homogeneous of degree k if,

for any number s where $s > 0$

$$s^k Y = f(sx_1, sx_2, \cdots, sx_n)$$

Cobb-Douglas Function

We have the Cobb - Douglas Production Function

$Y = \delta K^{\alpha} L^{\beta}$

Increase the inputs by a factor s

$Y_1 = \delta (sK)^{\alpha} (sL)^{\beta} = s^{\alpha+\beta} \delta K^{\alpha} L^{\beta} = s^{\alpha+\beta} Y$

Our Cobb - Douglas function is homogeneous of degree $\alpha + \beta$.

If $\alpha + \beta < 1$, we have decreasing returns to scale.

If $\alpha + \beta = 1$, we have constant returns to scale.

If $\alpha + \beta > 1$, we have increasing returns to scale.

Euler’s Theorem

- Homogeneity of degree 1 is often called **linear homogeneity**.
- An important property of homogeneous functions is given by **Euler’s Theorem**.

Euler's Theorem

For any multivariate function $y = f(x_1, x_2, \cdots, x_n)$

that is homogeneous of degree k,

$$ky = x_1 f_1(x_1, x_2, \cdots, x_n) + \cdots + x_n f_n(x_1, x_2, \cdots, x_n)$$

for any set of values $(x_1, x_2, \cdots, x_n)$, where $f_i(x_1, x_2, \cdots, x_n)$ is the partial derivative of the function with respect to its ith argument.

Proof Euler's Theorem

Definition homogeneou s function $s^K y = f(sx_1, sx_2, \cdots, sx_n)$

Take the partial derivative of the above with respect to s

$$ks^{k-1} y = x_1 f_1(sx_1, sx_2, \cdots, sx_n) + \cdots + x_n f_n(sx_1, sx_2, \cdots, sx_n)$$

Letting $s = 1$, we get Euler' s Theorem

$$ky = x_1 f_1(x_1, x_2, \cdots, x_n) + \cdots + x_n f_n(x_1, x_2, \cdots, x_n)$$

The converse of this theorem holds. If the above is true, then the original function is homogeneou s of degree k.

Division of National Income

Suppose that the national production function is

$Y = \alpha K^{\beta} L^{1-\beta}$ which is homogeneous of degree 1, therefore

$$\mathrm{Y} = \frac{\partial Y}{\partial K} K + \frac{\partial Y}{\partial L} L$$

Now under perfect competition, capital and labor are paid respectively their real return and real wage. This implies

$$wL = \frac{\partial Y}{\partial L} L = \left[(1-\beta)\alpha K^{\beta} L^{-\beta}\right] L = (1-\beta) Y$$

$$and \quad rK = \frac{\partial Y}{\partial K} K = \left[\beta \alpha K^{\beta-1} L^{1-\beta}\right] K = \beta Y.$$

$$Hence, Y = rK + wL = \beta Y + (1-\beta) Y$$

Properties of Marginal Products

For our national income accounting production function,

$$\frac{\partial Y}{\partial K} = \beta\alpha K^{\beta-1}L^{1-\beta} \text{ which is homogeneous of degree zero.}$$

Likewise for the marginal product of Labor,

$$\frac{\partial Y}{\partial L} = (1-\beta)\alpha K^{\beta}L^{-\beta}. \text{ We can write the marginal products as}$$

$$\frac{\partial Y}{\partial K} = \beta\alpha K^{\beta-1}L^{1-\beta} = \alpha\beta\left(\frac{L}{K}\right)^{1-\beta} \quad \textit{and}$$

$$\frac{\partial Y}{\partial L} = (1-\beta)\alpha K^{\beta}L^{-\beta} = \alpha(1-\beta)\left(\frac{K}{L}\right)^{\beta}$$

Arguments of Functions that are Homogeneous degree zero

Any function $f(x_1, x_2, \cdots, x_i, \cdots, x_n)$ that is homogeneous of degree zero can be written as

$$f\left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \cdots, 1, \cdots, \frac{x_n}{x_i}\right) \text{ for any } i = 1, 2, \ldots, n.$$

Proof : Since the function is homogeneous of degree 0,

$$s^0 f(x_1, x_2, \cdots, x_i, \cdots, x_n) = f(sx_1, sx_2, \cdots, sx_i, \cdots, sx_n)$$

$$Let\ s = \frac{1}{x_i},\ then$$

$$f(x_1, x_2, \cdots, x_i, \cdots, x_n) = f\left(\frac{x_1}{x_i}, \frac{x_2}{x_i}, \cdots, 1, \cdots, \frac{x_n}{x_i}\right)$$

QED

First Partial Derivatives of Homogeneous Functions

If the function, $f\left(x_1, x_2, \cdots, x_n\right)$ is homogeneous of degree k, then each of ists first partial derivatives

$$f_i = \frac{\partial f\left(x_1, x_2, \cdots, x_n\right)}{\partial x_i}$$ for any $\mathrm{i} = 1, 2, \cdots, \mathrm{n}$, is homogeneous of degree k-1.

Proof of previous slide

$$We\ know\ \ f\left(sx_1, sx_2, \cdots, sx_n\right) = s^k f\left(x_1, x_2, \cdots, x_n\right)$$

$$\frac{\partial f\left(sx_1, sx_2, \cdots, sx_n\right)}{\partial x_i} = \frac{\partial f\left(sx_1, sx_2, \cdots, sx_n\right)}{\partial\left(sx_i\right)} \cdot \frac{d\left(sx_i\right)}{dx_i}$$

$$= sf_i\left(sx_1, sx_2, \cdots, sx_n\right) \quad and$$

$$\frac{\partial s^k f\left(x_1, x_2, \cdots, x_n\right)}{\partial x_i} = s^k f_i\left(x_1, x_2, \cdots, x_n\right) \text{setting the two equal}$$

$$sf_i\left(sx_1, sx_2, \cdots, sx_n\right) = s^k f_i\left(x_1, x_2, \cdots, x_n\right) or$$

$$f_i\left(sx_1, sx_2, \cdots, sx_n\right) = s^{k-1} f_i\left(x_1, x_2, \cdots, x_n\right)$$

Which implies the derivative is homogeneous of degree *k-1*.

Homothetic function

A homothetic function is a montonic transformation of a homogeneous functin. This if $y = f(x_1, x_2, \cdots, x_n)$ is a homogeneous function, then $z = g(y)$ is a homothetic function if the function $g(y)$ is strictly monotonic, that is $g'(y) > 0$ for all y or if $g'(y) < 0$ for all y.

Example homothetic function

$let\ y = x^{\alpha} z^{\beta}$ which is homogeneous of degree $\alpha + \beta$.

Let $w = \ln(y) = \alpha \ln(x) + \beta \ln(z)$

$now\ \alpha \ln(sx) + \beta \ln(sz) = \alpha \ln(x) + \beta \ln(z) + (\alpha + \beta)\ln(s)$

$= w + (\alpha + \beta)\ln(s) \neq s^{k} w$

except when are original function is homogeneous of

degree ?.

Therefore, while homogeneous functions are homothetic,

not all homothetic functions are homogeneous.