# Economics 2301

Lecture 29

Multivariate Calculus

# Total Differential

The total differential of the multivariate function

$y = f(x_1, x_2, \cdots, x_n)$ evaluated at the point

$\left(x_1^0, x_2^0, \cdots, x_n^0\right)$ is

$$dy = f_1\left(x_1^0, x_2^0, \cdots, x_n^0\right)dx_1 + \cdots + f_n\left(x_1^0, x_2^0, \cdots, x_n^0\right)dx_n$$

where $f_i\left(x_1^0, x_2^0, \cdots, x_n^0\right)$ represents the partial derivative of the function $f\left(x_1, x_2, \cdots, x_n\right)$ with respect to the ith argument, evaluated at the point $\left(x_1^0, x_2^0, \cdots, x_n^0\right)$.

# Example

Consider the utility function, $U = 5\ln(x)\ln(y)$.

Marginal Utilities: $U_x = \dfrac{5\ln(y)}{x}$ and $U_y = \dfrac{5\ln(x)}{y}$

Diminishing marginal Utility:

$$U_{xx} = -\frac{5\ln(y)}{x^2} < 0 \text{ and } U_{yy} = -\frac{5\ln(x)}{y^2} < 0$$

*Let* $x = 2$ *and* $y = 3$. *Also let* $dx = 0.1$ *and* $dy = 0.1$, *then*

$$dU = \frac{5\ln(y)}{x}dx + \frac{5\ln(x)}{y}dy = \frac{5\ln(3)}{2}(0.1) + \frac{5\ln(2)}{3}(0.1)$$

$$= \frac{5(1.0986)}{2}(0.1) + \frac{5(0.6931)}{3}(0.1) = 0.275 + 0.116 = 0.391$$

Actual Change = 0.3896, The % error = 0.36%

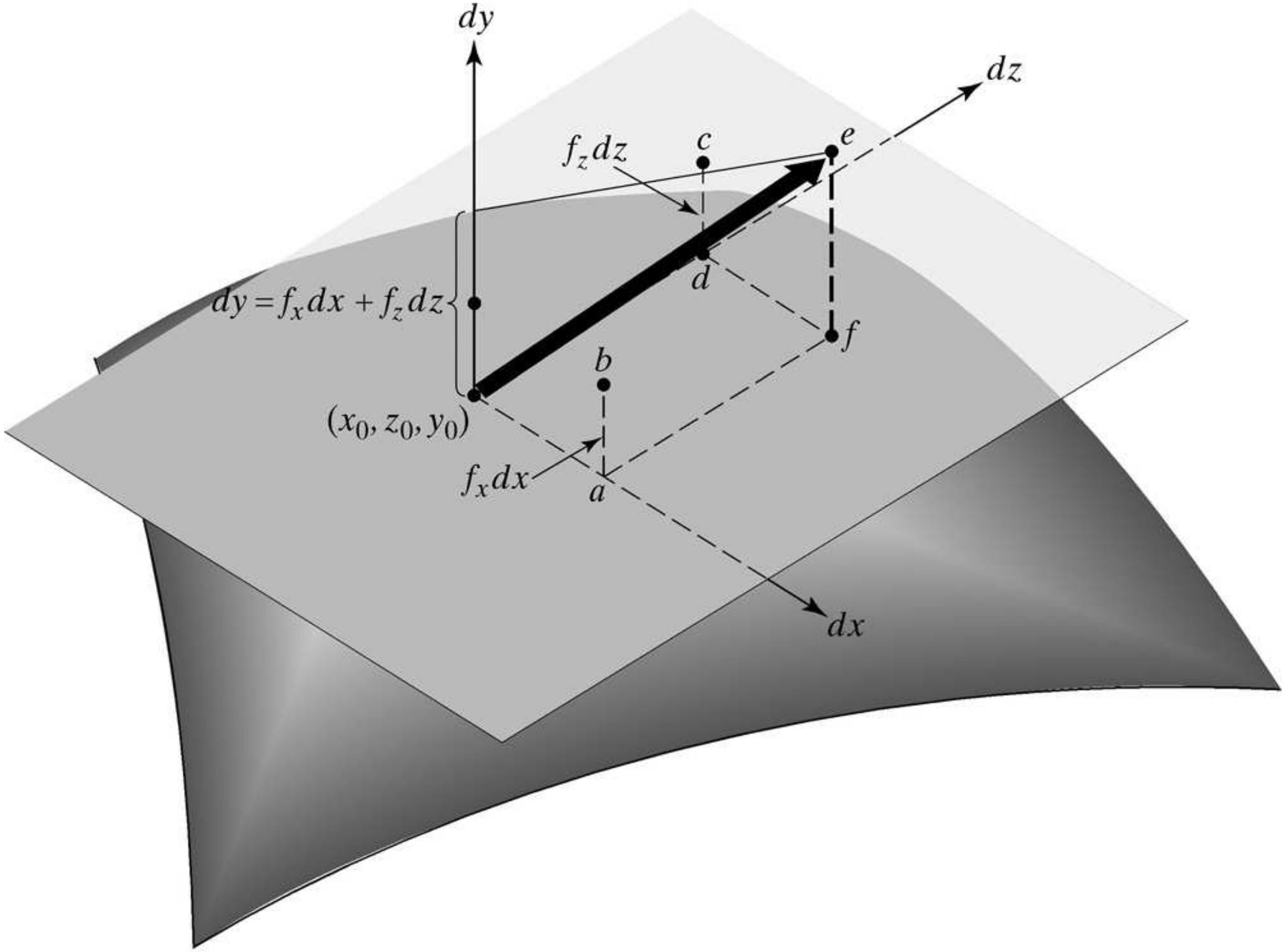
# Geometric Interpretation

In a multivariate function with two arguments x and z, the differential at point $(x_0,z_0,y_0)$ can be interpreted as describing points on the two-dimensional plane that passes through $(x_0,z_0,y_0)$ and is tangent to the surface of the original multivariate function. Points on the tangent plane satisfy the differential

$$dy=f_x(x,z)dx+f_z(x,z)dz$$

This tangent plane is illustrated in Figure 8.4. The slope of a slice of this tangent plane along the dx axis is $f_x(x_0,z_0)$ and the slope of a slice along the dz axis is $f_z(x_0,z_0)$

# **Figure 8.4** Differential of a Multivariate Equation at Point ($x_0$, $z_0$, $y_0$)



# Implicit Functions

- Implicit function combines the dependent variable and the independent variables in a form like $F(y, x_1, x_2, \ldots, x_n) = k$. Often $k=0$.
- Implicit functions are often used in the context of level curves, which show how the arguments of a function are related to a particular level of a variable.
- An indifference curve and an isoquant are particular types of level curves.

# Implicit Function Theorem

For an implicit function

$F(y, x_1, x_2, \cdots, x_n) = k$, for which $k$ is a constant, which is defined at the point $\left(y^0, x_1^0, x_2^0, \cdots, x_n^0\right)$ and which has continuous first partial derivatives at that point with $F_y\left(y^0, x_1^0, x_2^0, \cdots, x_n^0\right) \neq 0$ there is a function $y = f\left(x_1, x_2, \cdots, x_n\right)$ define in the neighborhood of $\left(y^0, x_1^0, x_2^0, \cdots, x_n^0\right)$ corresponding to $F(y, x_1, x_2, \cdots, x_n) = k$ such that

i. $F\left(f\left(x_1^0, x_2^0, \cdots, x_n^0\right), x_1^0, x_2^0, \cdots, x_n^0\right) = k$

$ii.$ $y^0 = f\left(x_1^0, x_2^0, \cdots, x_n^0\right)$

$iii.$ $f_i\left(x_1^0, x_2^0, \cdots, x_n^0\right) = -\dfrac{F_{x_i}\left(y^0, x_1^0, x_2^0, \cdots, x_n^0\right)}{F_y\left(y^0, x_1^0, x_2^0, \cdots, x_n^0\right)}$

# Example

Our previous utility function was $U = 5\ln(x)\ln(y)$.

Suppose utility is at the level 10 (i.e. $5\ln(x)\ln(y) = 10$).

By the implicit function theorem,

$$\frac{dy}{dx} = -\frac{5\ln(y)}{x} \Big/ \frac{5\ln(x)}{y} = -\frac{\ln(y)}{\ln(x)} \cdot \frac{y}{x}$$

Our implicit function can be written as

$$\ln(y) = \frac{2}{\ln(x)} \rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{2}{(\ln(x))^2} \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{(\ln(x))^2} \cdot \frac{y}{x}$$

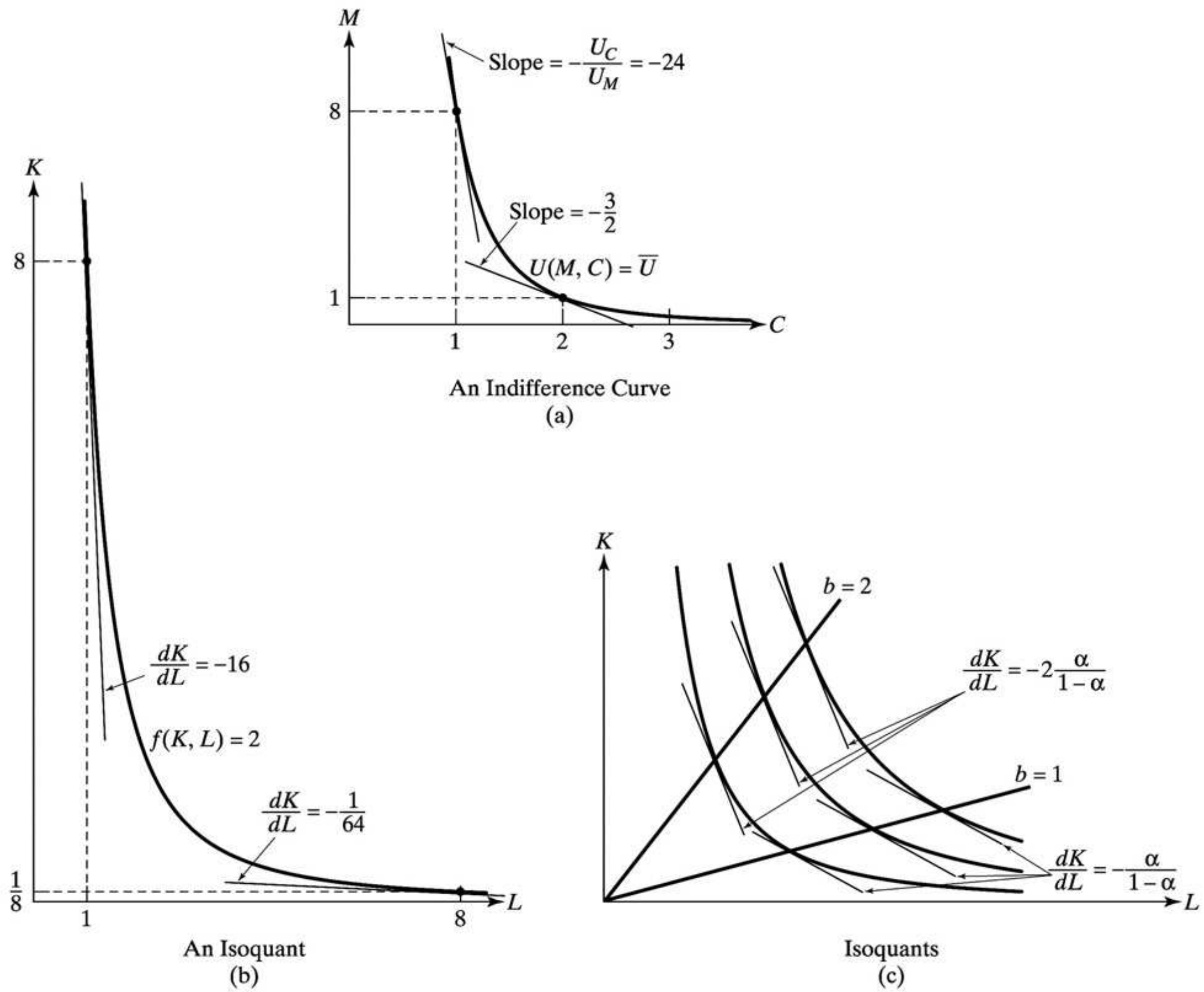
Substituting the solution for ln(y) into the first version, we get

$$\frac{dy}{dx} = -\frac{\ln(y)}{\ln(x)} \cdot \frac{y}{x} = -\frac{2}{(\ln(x))^2} \cdot \frac{y}{x}$$

# Our Example Continued

If we review our derivative derived by the implicit function theorem, we discover that the slope of our indifference curve is the negative of the ratio of the marginal utility of $x$ to the marginal utility of $y$.

# Figure 8.6 An Indifference Curve and Isoquants



# Cobb-Douglas Production Function

Recall the Cobb - Douglas Production function we use in our national output example.

$$y = \alpha K^{\beta} L^{1-\beta}$$

Now we can define an isoquant at the level of output, $y^0$.

$\alpha K^{\beta} L^{1-\beta} = y^o$, a line tangent to the isoquant will have slope

$$\frac{dK}{dL} = -\left(\left(1-\beta\right)\alpha K^{\beta} L^{-\beta}\right) / \left(\beta \alpha K^{\beta-1} L^{1-\beta}\right) = -\frac{1-\beta}{\beta} \cdot \left(\frac{K}{L}\right)$$

# Cobb-Douglas Continued

Note that an array from the origin in the K - L space has the formula : K = bL. Where b is the slope of the array.

Given the slope of our isoquant,

$\frac{dK}{dL} = -\frac{1-\beta}{\beta} \cdot \left(\frac{K}{L}\right)$ then along the array K = bL, our slope is the same for all isoquants

$$\frac{dK}{dL} = -\frac{1-\beta}{\beta} \cdot \left(\frac{K}{L}\right) = -\frac{1-\beta}{\beta} \cdot \left(\frac{bL}{L}\right) = -\frac{1-\beta}{\beta} \cdot b$$

The slope does not depend on the scale.

# Homogeneous Functions

The slope of the level curves of any homogeneous function are constant along any ray from the origin. The slope of a level curve of a function $f(x_1,x_2)$ is

$$\frac{dx_2}{dx_1} = -\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}$$

Recall if function homogeneous degree $k$, then partial derivatives are homogeneous of degree $k$-$1$.

$f_i(sx_1,sx_2) = s^{k-1} f_i(x_1,x_2)$, thus along an array from the origin

$$\frac{dx_2}{dx_1} = -\frac{f_1(sx_1,sx_2)}{f_2(sx_1,sx_2)} = -\frac{s^{k-1} f_1(x_1,x_2)}{s^{k-1} f_2(x_1,x_2)} = -\frac{f_1(x_1,x_2)}{f_2(x_1,x_2)}$$

# Homothetic Function

For a homogeneous function, suppose we define a homothetic function $z = g(y) = g(f(x_1, x_2))$ where $g'(y) > 0$ or $g'(y) < 0$ for all $y$.

$$\frac{dx_2}{dx_1} = -\frac{g'(y) f_1(sx_1, sx_2)}{g'(y) f_2(sx_1, sx_2)} = -\frac{f_1(sx_1, sx_2)}{f_2(sx_1, sx_2)} = -\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$$

This show that the slope of the level curves of any homothetic function, a class of functions that includes, but is not limited to, homogeneous functions, is not altered by a proportional scaling of all of its arguments.

# Example

Consider the utility function, $U = \left[\delta X^{-\rho} + (1-\delta)Y^{-\rho}\right]^{-\frac{1}{\rho}}$,

suppose we difine the homothetic function, z = ln(U).

Now the slope of our indifference curve is

$$\frac{dY}{dX} = -\frac{(1/U)\cdot\left(-\frac{1}{\rho}\right)\left[\delta X^{-\rho} + (1-\delta)Y^{-\rho}\right]^{-\frac{1}{\rho}-1}\cdot\left(-\rho\delta X^{-\rho-1}\right)}{(1/U)\cdot\left(-\frac{1}{\rho}\right)\left[\delta X^{-\rho} + (1-\delta)Y^{-\rho}\right]^{-\frac{1}{\rho}-1}\cdot\left(-\rho(1-\delta)Y^{-\rho-1}\right)}$$

$$= -\frac{\delta}{1-\delta}\left(\frac{Y}{X}\right)^{1+\rho}$$ Which is constant along the array Y = bX.