Economics 2301

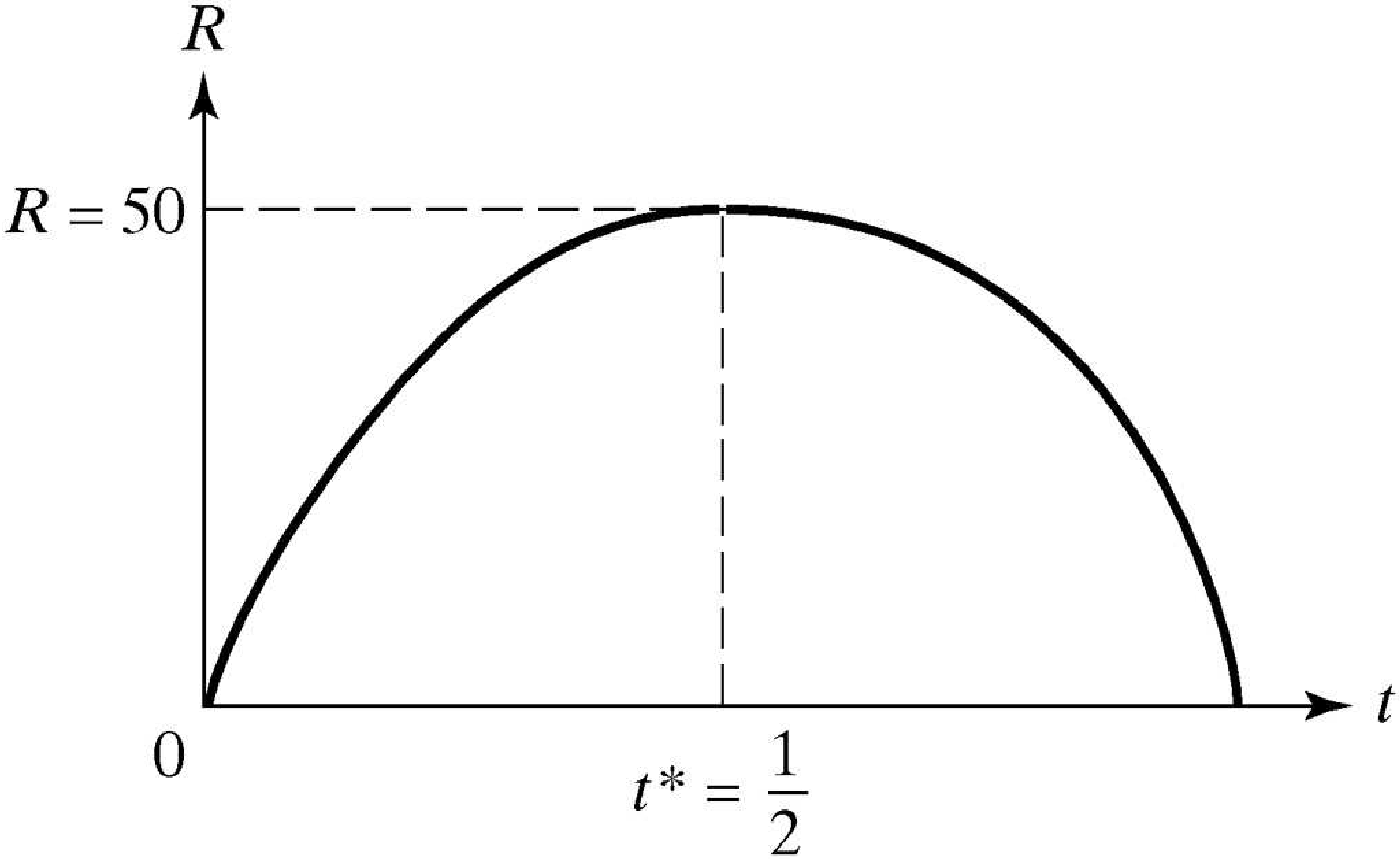
Lecture 30

Univariate Optimization

Identifying and Characterizing Extreme Values

- Extreme value of a univariate function can occur either within the interval over which the function is defined or at its endpoints.
- Strictly monotonic functions do not have extreme values within the interior of their domain, only at the endpoints.
- Nonmonotonic functions may have one or more extreme values within the interior of the domain.

Figure 9.1 Tax Rate and Tax Revenue



Stationary Point

We say that x^* is a stationary point of a differentiable function $f(x)$ if

$$f'(x^*) = 0.$$

First Order Condition

If the function $y = f(x)$ is everywhere differentiable on an interval and reaches a maximum or a minimum at the point x^* within that interval, then x^* is a stationary point; that is $f'(x^*) = 0$.

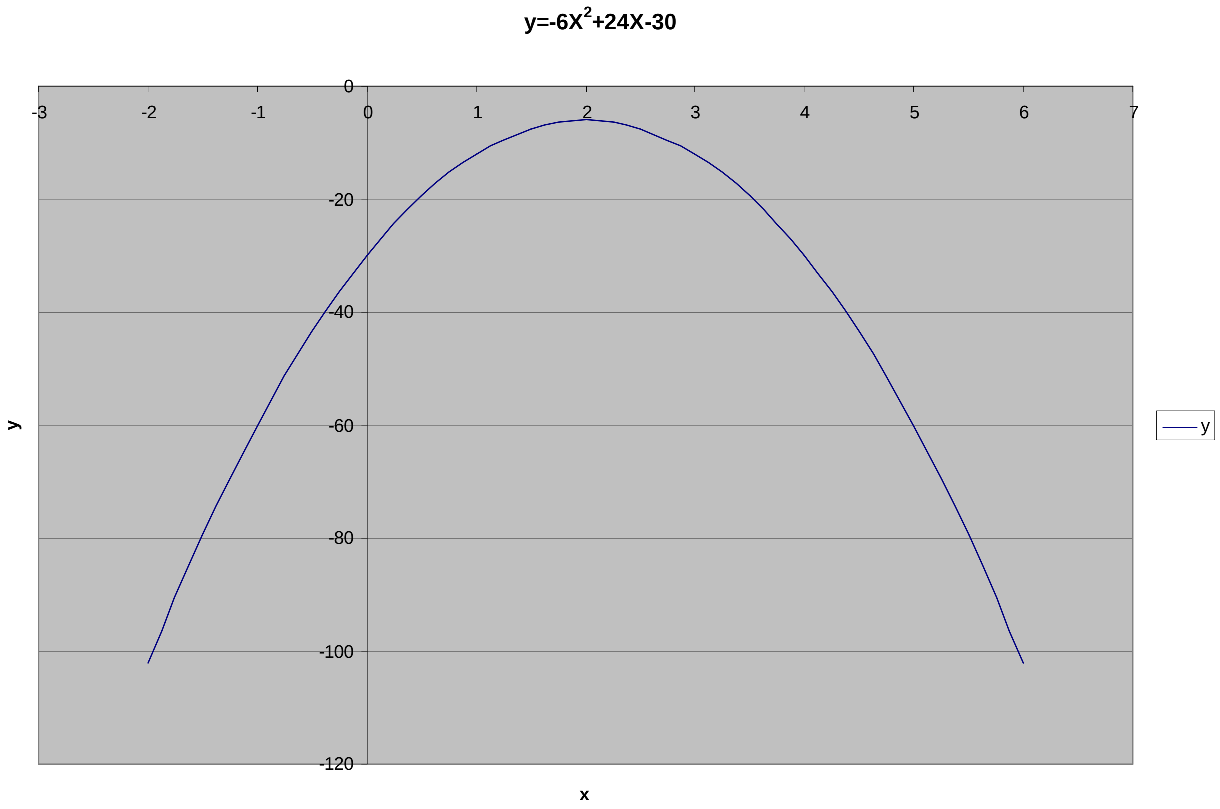
Example 1

$$Let \quad y = -6x^2 + 24x - 30$$

This function has a local maximum at $x^* = 2$.

$$\text{Note}: \frac{dy}{dx} = -12x + 24 = 0 \; at \; x^* = 2.$$

Graph of Example 1



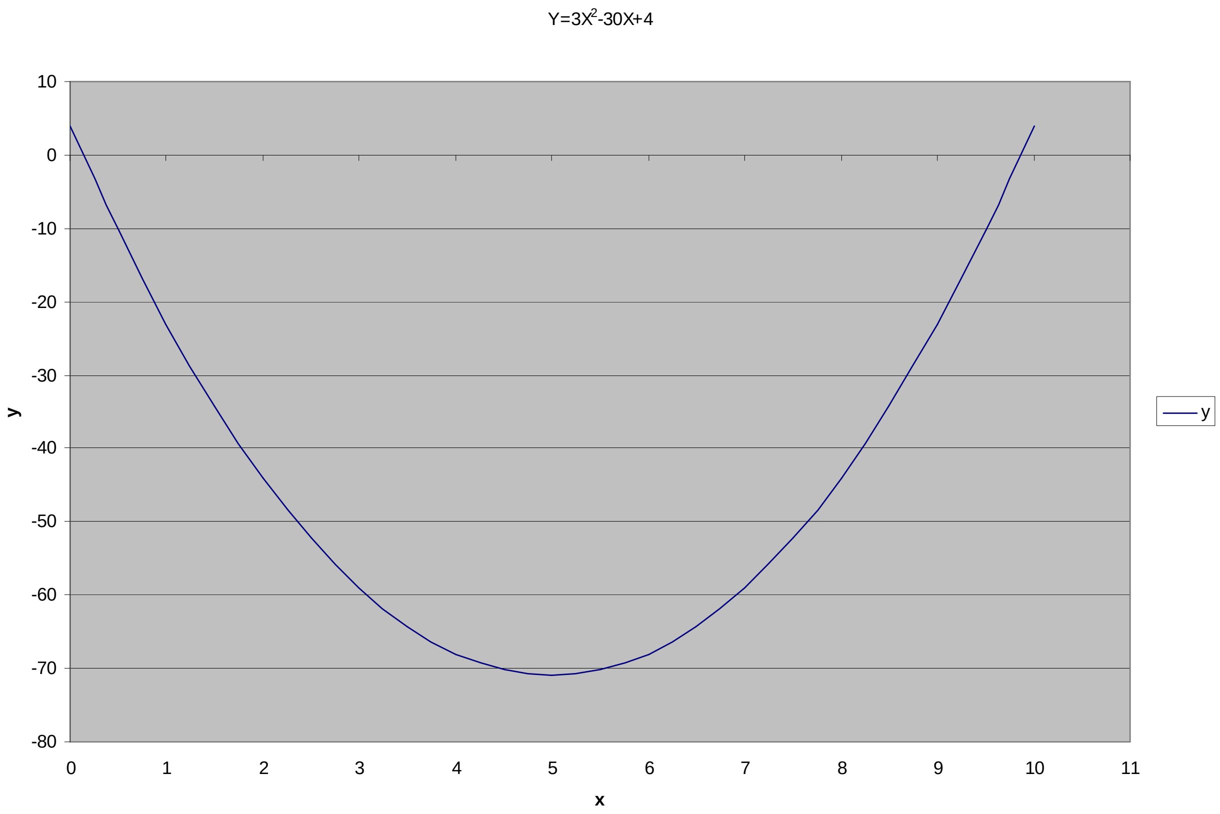
Example 2

$$Let\ y = 3x^2 - 30x + 4.$$

This function reaches a minimum at $x^* = 5$.

$$\frac{dy}{dx} = 6x - 30 = 0\ at\ x^* = 5.$$

Graph of Example 2



Function with 2 extreme values

$$Let\ y = 0.2X^3 - 5X^2 + 15X - 4$$

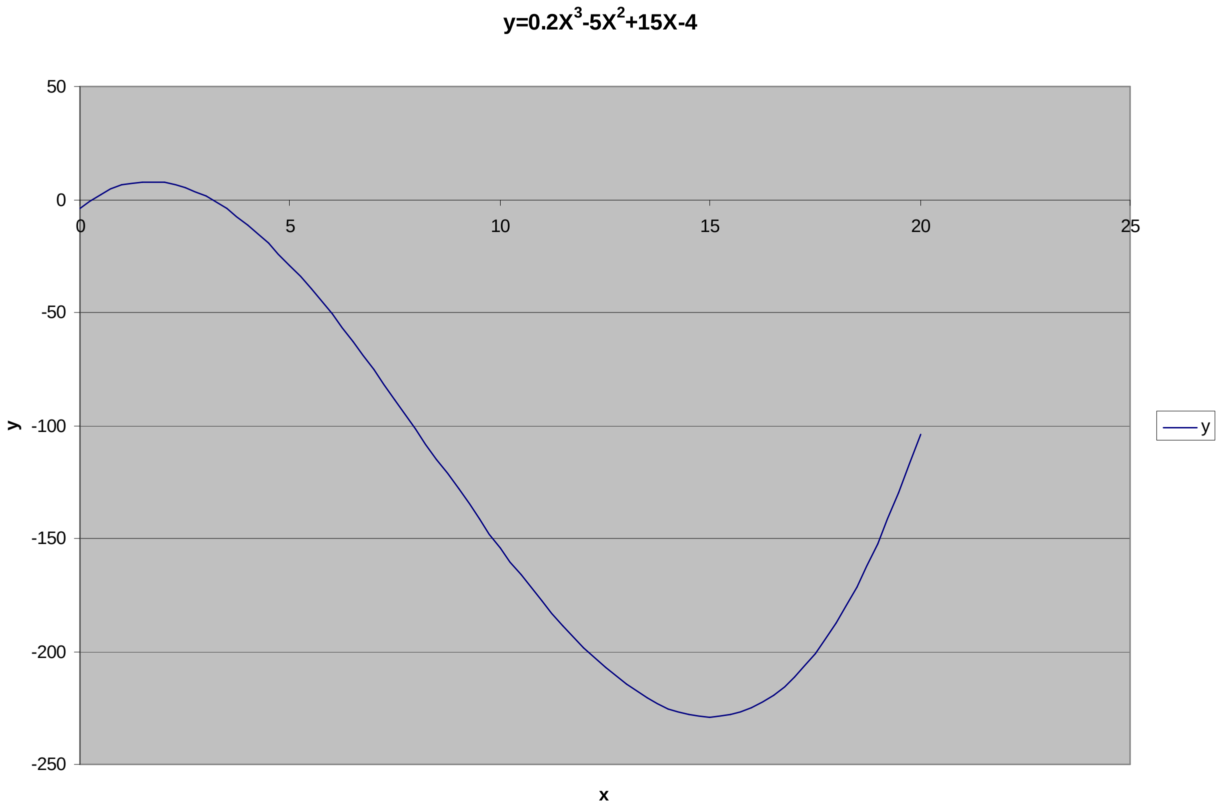
$$\text{This function has two extreme values at X*} = 1.67 \text{ and } 15,$$

$$\frac{dy}{dx} = .6X^2 - 10X + 15 = 0$$

$$X* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{10^2 - 4 \cdot 0.6 \cdot 15}}{2 \cdot 0.6}$$

$$= \frac{10 \pm \sqrt{100 - 36}}{1.2} = \frac{10 \pm \sqrt{64}}{1.2} = \frac{10 \pm 8}{1.2} = 1.67;15$$

Graph of Example 3



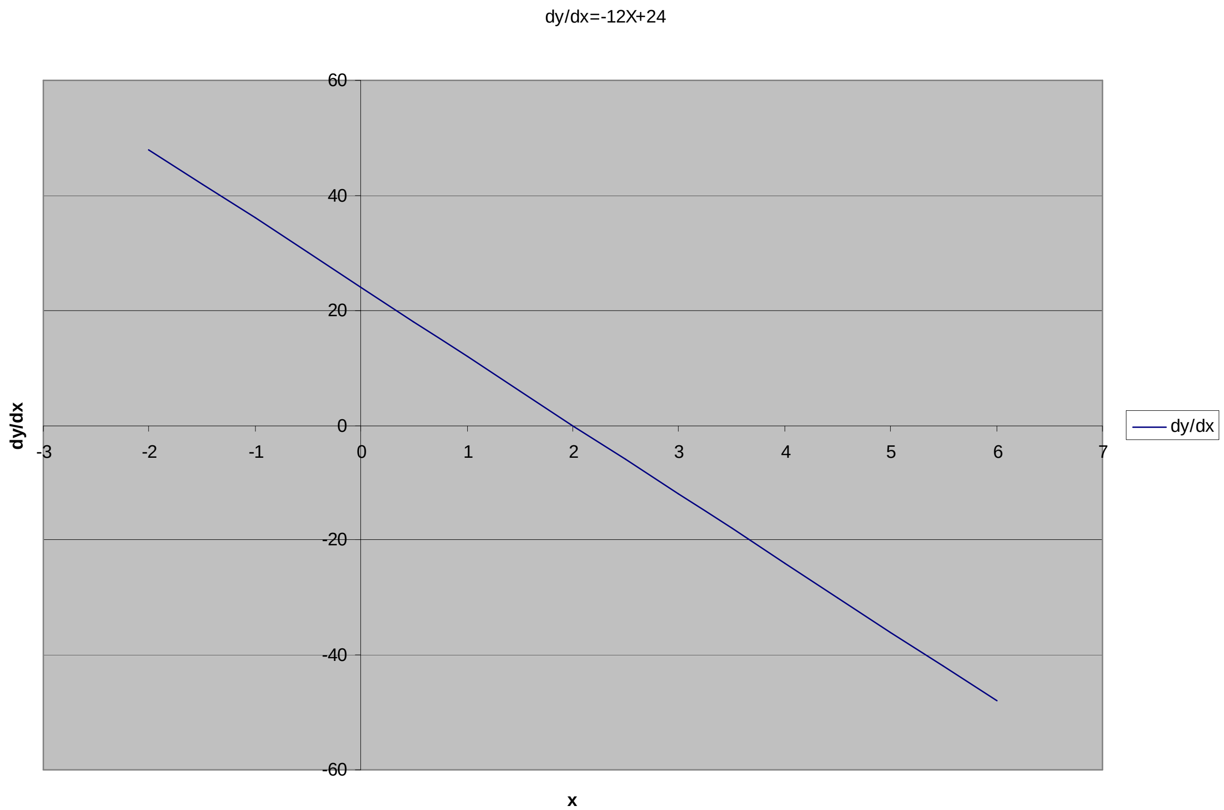
Critical Point and First-Order Conditions

- We say that x^* is a **critical point** of a function $f(x)$ if $f'(x^*)=0$.
- **First Order Condition**: If the function $f(x)$ achieves a maximum or a minimum at the point x^* within an interval on which it is defined, then x^* is a critical point of that function.

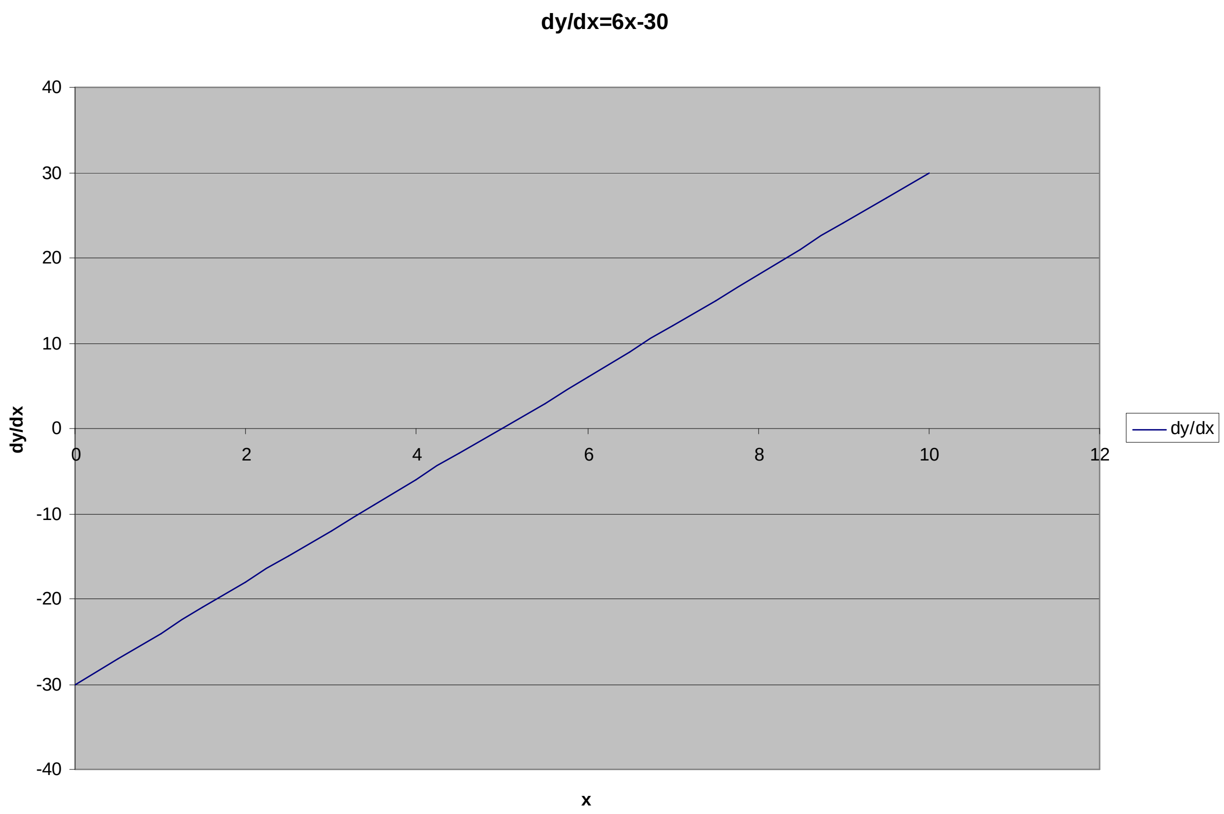
Characterizing Stationary Points

- **Global Maximum**: If a function a function *f(x)* that is everywhere differentiable has the stationary point x^*, this this stationary point represents a global maximum if $f'(x) \geq 0$ for all $x \leq x^*$, and $f'(x) \leq 0$ for all $x \geq x^*$.
- **Global Minimum:** If a function a function *f(x)* that is everywhere differentiable has the stationary point x^*, this this stationary point represents a global minimum if $f'(x) \leq 0$ for all $x \leq x^*$, and $f'(x) \geq 0$ for all $x \geq x^*$.

Graph of first derivative of Example 1



Graph of first derivative of Example 2



Local Maximum

- If a function $f(x)$ that is everywhere differentiable has an interior local maximum at the stationary point x^*, then, throughout some interval to the left of the stationary point, (m,x^*), $f'(x) \geq 0$ and, throughout some interval to the right of the stationary point, (x^*,n), $f'(x) \leq 0$.

Local Minimum

- If a function $f(x)$ that is everywhere differentiable has an interior local minimum at the stationary point x^*, then, throughout some interval to the left of the stationary point, (m,x^*), $f'(x) \leq 0$ and, throughout some interval to the right of the stationary point, (x^*,n), $f'(x) \geq 0$.

Graph of first derivative of example 3