Economics 2301

Lecture 30

Univariate Optimization
Identifying and Characterizing Extreme Values

- Extreme value of a univariate function can occur either within the interval over which the function is defined or at its endpoints.
- Strictly monotonic functions do not have extreme values within the interior of their domain, only at the endpoints.
- Nonmonotonic functions may have one or more extreme values within the interior of the domain.
Figure 9.1  Tax Rate and Tax Revenue

\[ R = 50 \]

\[ t^* = \frac{1}{2} \]
We say that $x^*$ is a stationary point of a differentiable function $f(x)$ if $f'(x^*) = 0$. 
First Order Condition

If the function \( y = f(x) \) is everywhere differentiable on an interval and reaches a maximum or a minimum at the point \( x^* \) within that interval, then \( x^* \) is a stationary point; that is \( f'(x^*) = 0 \).
Example 1

Let \( y = -6x^2 + 24x - 30 \)

This function has a local maximum at \( x^* = 2 \).

Note: \( \frac{dy}{dx} = -12x + 24 = 0 \) at \( x^* = 2 \).
Graph of Example 1

\[ y = 6x^2 + 24x - 30 \]
Example 2

Let \( y = 3x^2 - 30x + 4 \).

This function reaches a minimum at \( x^* = 5 \).

\[
\frac{dy}{dx} = 6x - 30 = 0 \text{ at } x^* = 5.
\]
Graph of Example 2

$Y = 30X^2 - 30X + 4$
Function with 2 extreme values

Let \( y = 0.2X^3 - 5X^2 + 15X - 4 \)
This function has two extreme values at \( X^* = 1.67 \) and 15,

\[
\frac{dy}{dx} = .6X^2 - 10X + 15 = 0
\]

\[
X^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 - 4 \cdot 0.6 \cdot 15}}{2 \cdot 0.6}
\]

\[
= \frac{10 \pm \sqrt{100 - 36}}{1.2} = \frac{10 \pm \sqrt{64}}{1.2} = \frac{10 \pm 8}{1.2} = 1.67;15
\]
Graph of Example 3

\[ y = 0.2X^3 - 5X^2 + 15X - 4 \]
Critical Point and First-Order Conditions

- We say that $x^*$ is a **critical point** of a function $f(x)$ if $f'(x^*)=0$.

- **First Order Condition**: If the function $f(x)$ achieves a maximum or a minimum at the point $x^*$ within an interval on which it is defined, then $x^*$ is a critical point of that function.
Characterizing Stationary Points

- **Global Maximum**: If a function $f(x)$ that is everywhere differentiable has the stationary point $x^*$, this stationary point represents a global maximum if $f'(x) \geq 0$ for all $x \leq x^*$, and $f'(x) \leq 0$ for all $x \geq x^*$.

- **Global Minimum**: If a function $f(x)$ that is everywhere differentiable has the stationary point $x^*$, this stationary point represents a global minimum if $f'(x) \leq 0$ for all $x \leq x^*$, and $f'(x) \geq 0$ for all $x \geq x^*$. 
Graph of first derivative of Example 1

\[ \frac{dy}{dx} = -12x + 24 \]
Graph of first derivative of Example 2

\[ \frac{dy}{dx} = 6x - 30 \]
Local Maximum

- If a function $f(x)$ that is everywhere differentiable has an interior local maximum at the stationary point $x^*$, then, throughout some interval to the left of the stationary point, $(m, x^*)$, $f'(x) \geq 0$ and, throughout some interval to the right of the stationary point, $(x^*, n)$, $f'(x) \leq 0$. 
Local Minimum

- If a function \( f(x) \) that is everywhere differentiable has an interior local minimum at the stationary point \( x^* \), then, throughout some interval to the left of the stationary point, \((m, x^*)\), \( f'(x) \leq 0 \) and, throughout some interval to the right of the stationary point, \((x^*, n)\), \( f'(x) \geq 0 \).
Graph of first derivative of example 3

dy/dx = 0.6x^2 - 10x + 15