Economics 2301

Lecture 31

Univariate Optimization

Second-Order Conditions

- The second-order condition provides a sufficient condition but, as we will see, not a necessary condition, for characterizing a stationary point as a local maximum or a local minimum.

Second-Order Conditions

- **Local Maximum**: If the second derivative of the differentiable function $y=f(x)$ is negative when evaluated at a stationary point x^* (that is, $f''(x^*)<0$), then that stationary point represents a local maximum.
- **Local Minimum**: If the second derivative of the differentiable function $y=f(x)$ is positive when evaluated at a stationary point x^* (that is, $f''(x^*)>0$), then that stationary point represents a local minimum.

Second-Order Conditions for Our Examples

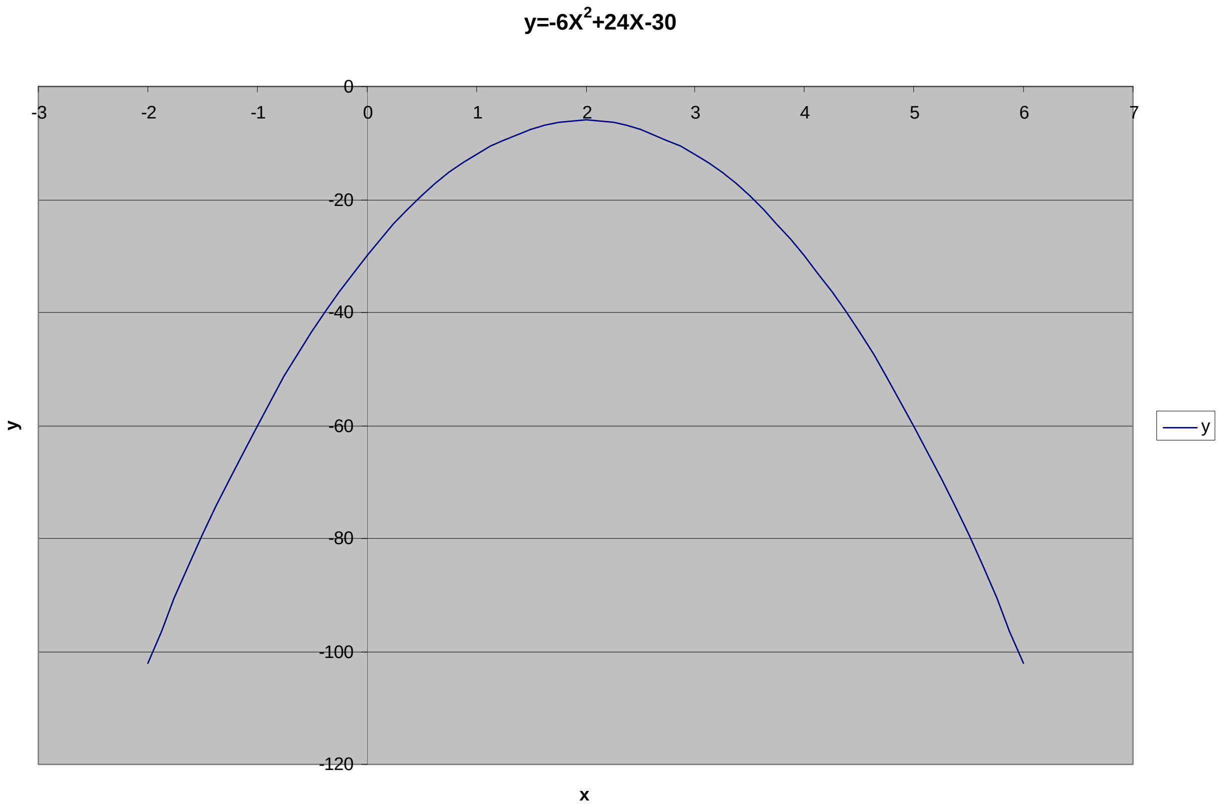
$$\text{Example 1}: y = -6x^2 + 24x - 30$$

$$\frac{dy}{dx} = -12x + 24; \quad \frac{d^2y}{dx^2} = -12 < 0, \therefore \text{a local maximum.}$$

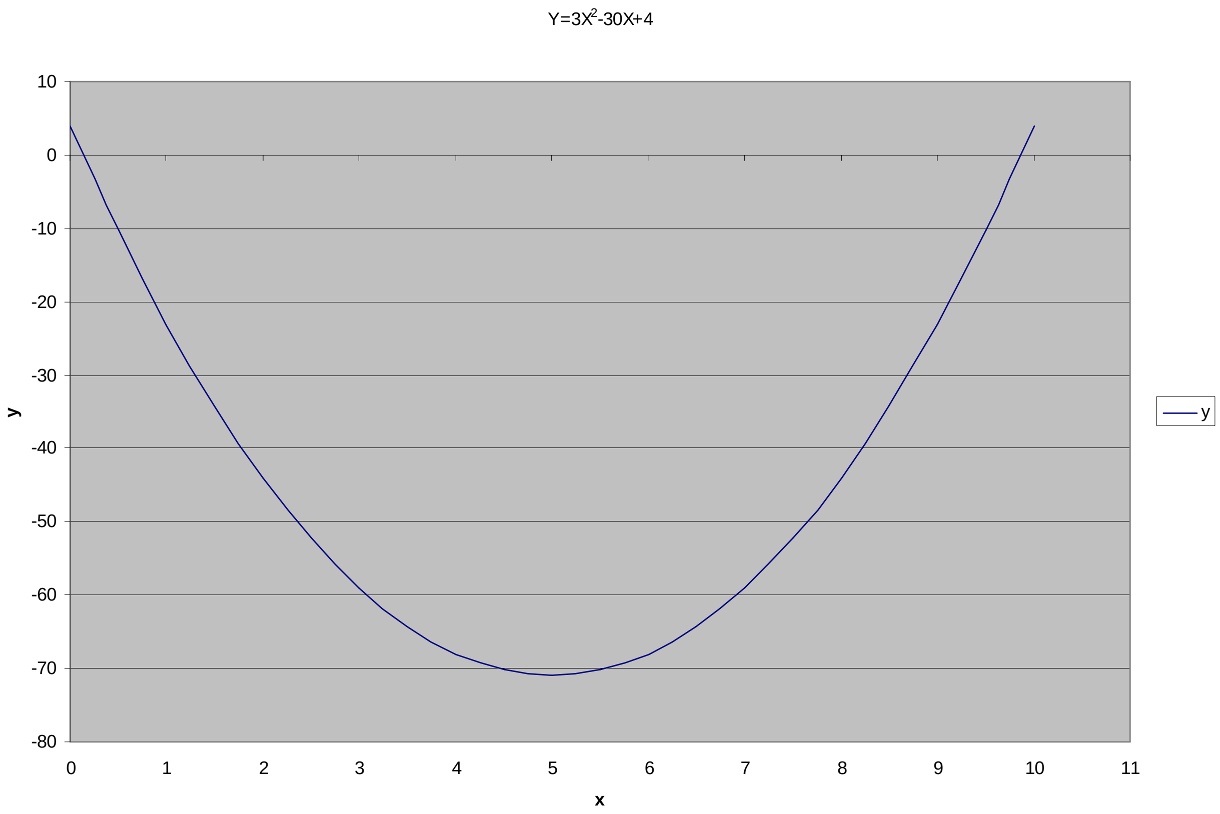
$$\text{Example 2}: y = 3x^2 - 30x + 4$$

$$\frac{dy}{dx} = 6x - 30; \quad \frac{d^2y}{dx^2} = 6 < 0, \therefore \text{a local minimum.}$$

Graph of Example 1



Graph of Example 2



Second-Order Conditions for Example 3

$$y = 0.2x^3 - 5x^2 + 15x - 4$$

$$\frac{dy}{dx} = 0.6x^2 - 10x + 15$$

$$\frac{d^2y}{dx^2} = 1.2x - 10$$

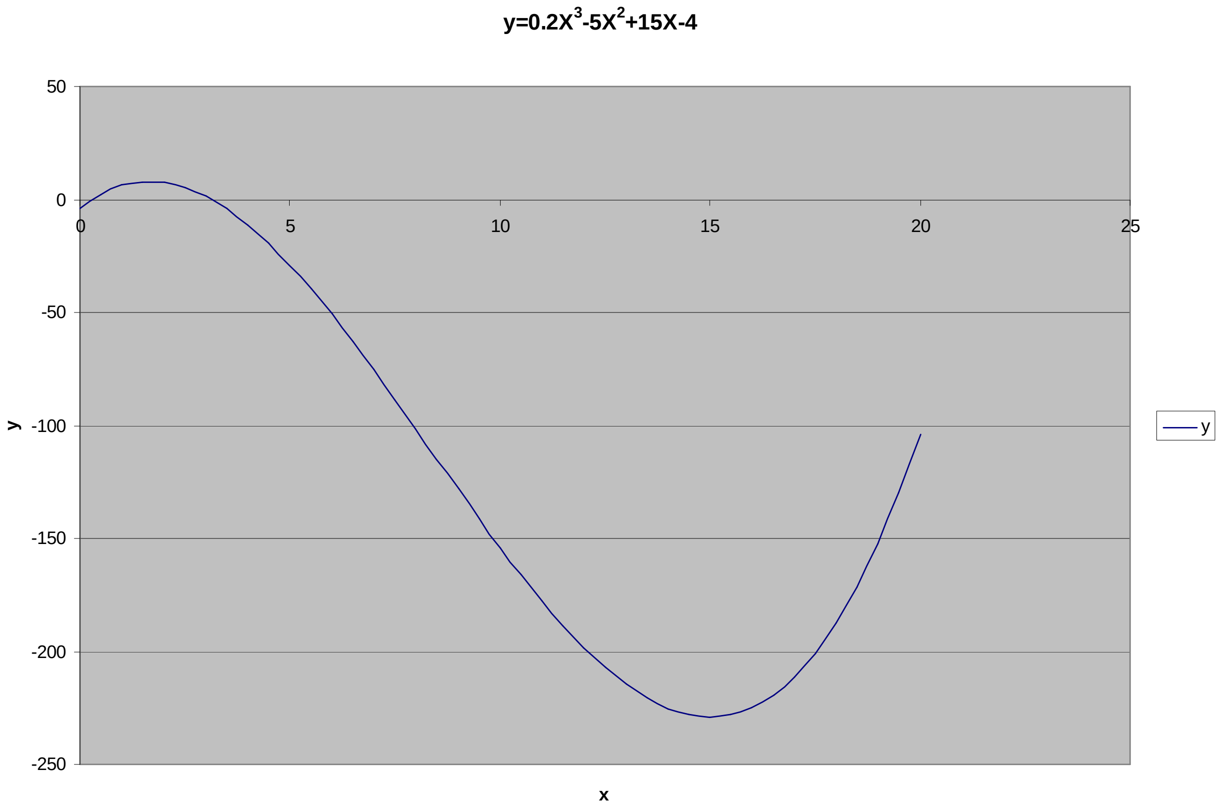
$$at\ x^* = 1.67; \frac{d^2y}{dx^2} = 1.2(1.67) - 10 = 2 - 10 = -8 < 0$$

$\therefore$ a local maximum.

$$at\ x^* = 15; \frac{d^2y}{dx^2} = 1.2(15) - 10 = 18 - 10 = 8 > 0$$

$\therefore$ a local minimum.

Graph of Example 3



Failure of Second-Order Condition

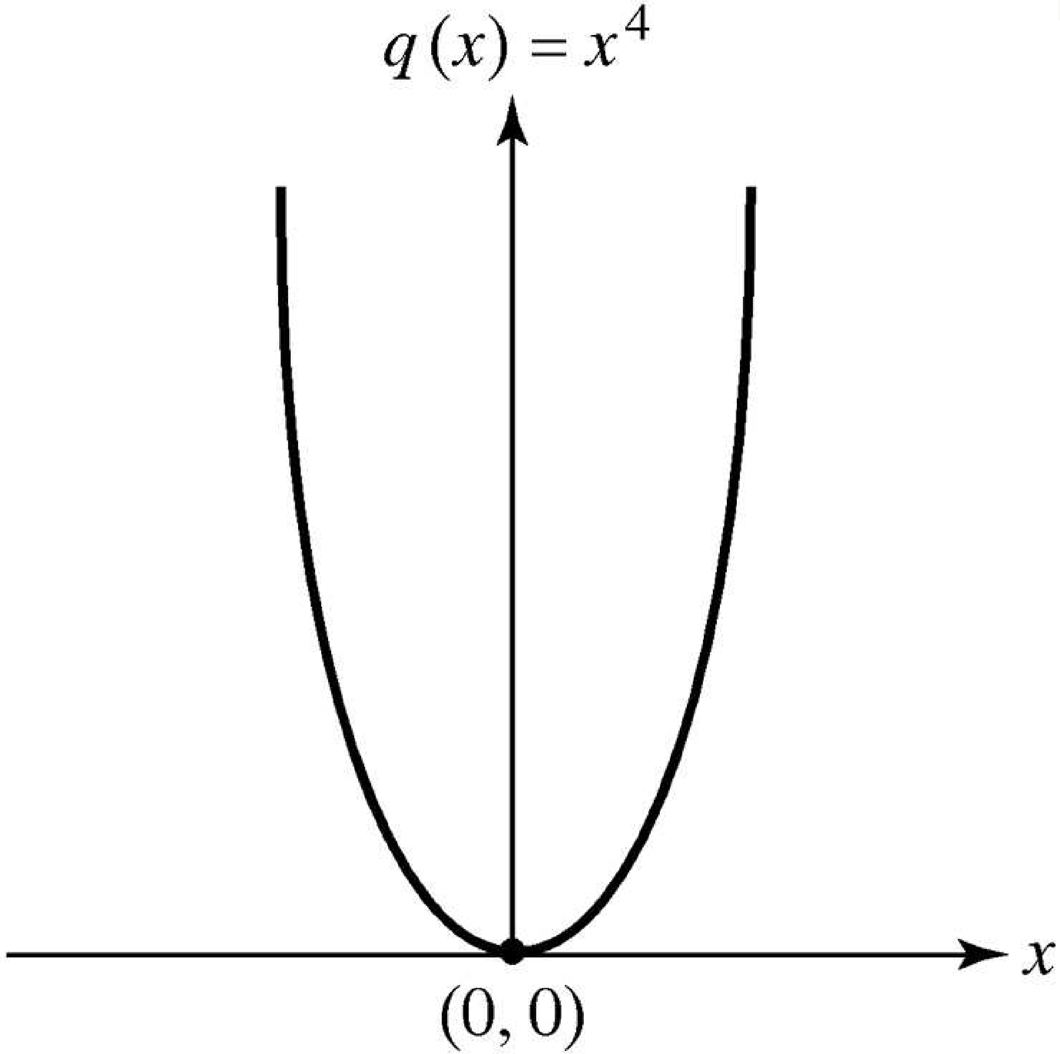
$$Let\ q(x) = x^4$$

$$q'(x) = 4x^3$$

$$\text{This equals 0 at x = 0.}$$

$$q''(x) = 12x^2 = 0\ at\ x = 0$$

Figure 9.6 Failure of the Second-Order Condition



Stationary Point of a strictly concave function

If the function $f(x)$ is strictly concave on the interval (m,n) and has the stationary point x^*, where $m<x^*<n$, the x^* is a local maximum in that interval. If a function is strictly concave everywhere, then it has, at most, one stationary point, and that stationary point is a global maximum.

Note that example 1 satisfies this condition.

Stationary Point of a Strictly Convex Function

If the function f(x) is strictly convex on the interval (m,n) and has the stationary point x^*, where $m<x^*<n$, the x^* is a local minimum in that interval. If a function is strictly convex everywhere, then it has, at most, one stationary point, and that stationary point is a global minimum.

Note that example 2 satisfies this condition.

Inflection Point

The twice-differentiable function f(x) has an inflection point at x* if and only if the sign of the second derivative switches from negative in some interval (m,x*) to positive in some interval (x*,n), in which case the function switches from concave to convex at x*, or the sign of the second derivative switches from positive in some interval (m,x*) to negative in some interval (x*,n), in which case the function switches from convex to concave at x*. Note that, in either case, m<x<n.

Example inflection point

For example 3, $y = 0.2x^3 - 5x^2 + 15x - 4$

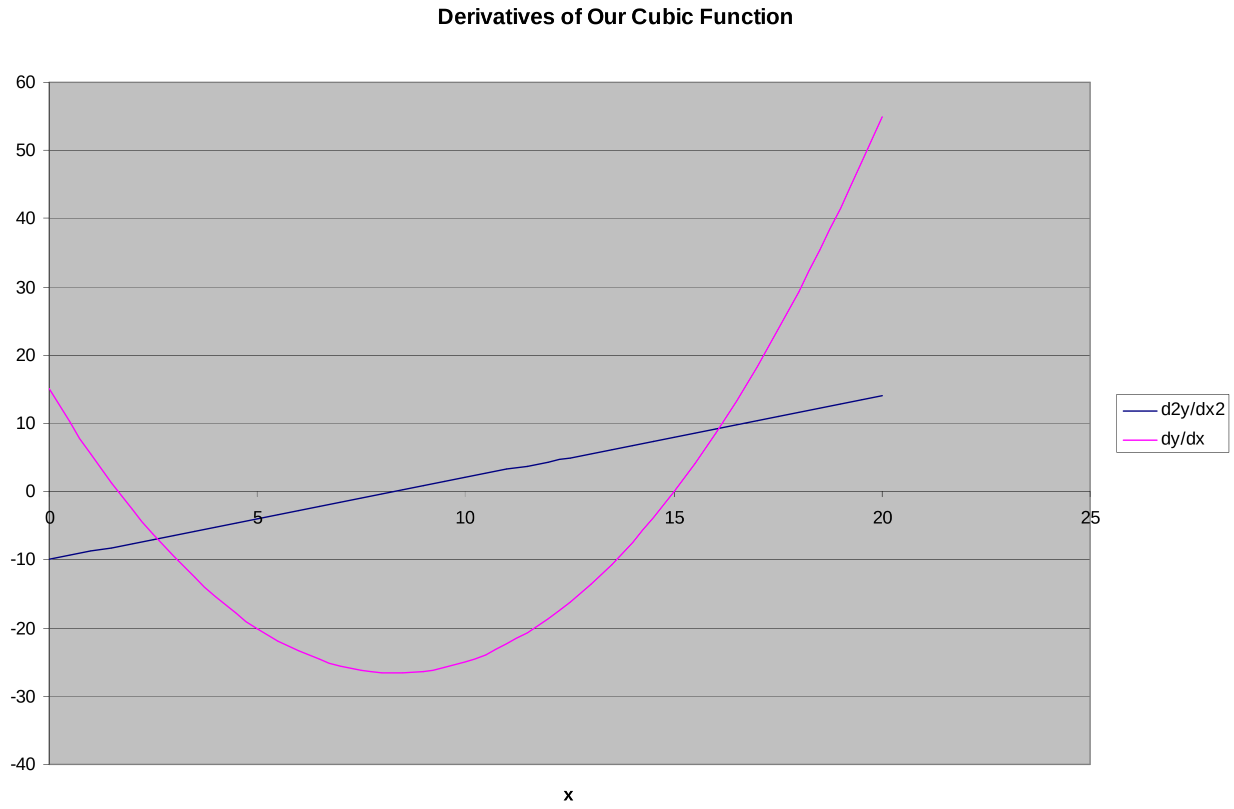
$$\frac{dy}{dx} = 0.6x^2 - 10x + 15$$

$$\frac{d^2y}{dx^2} = 1.2x - 10$$

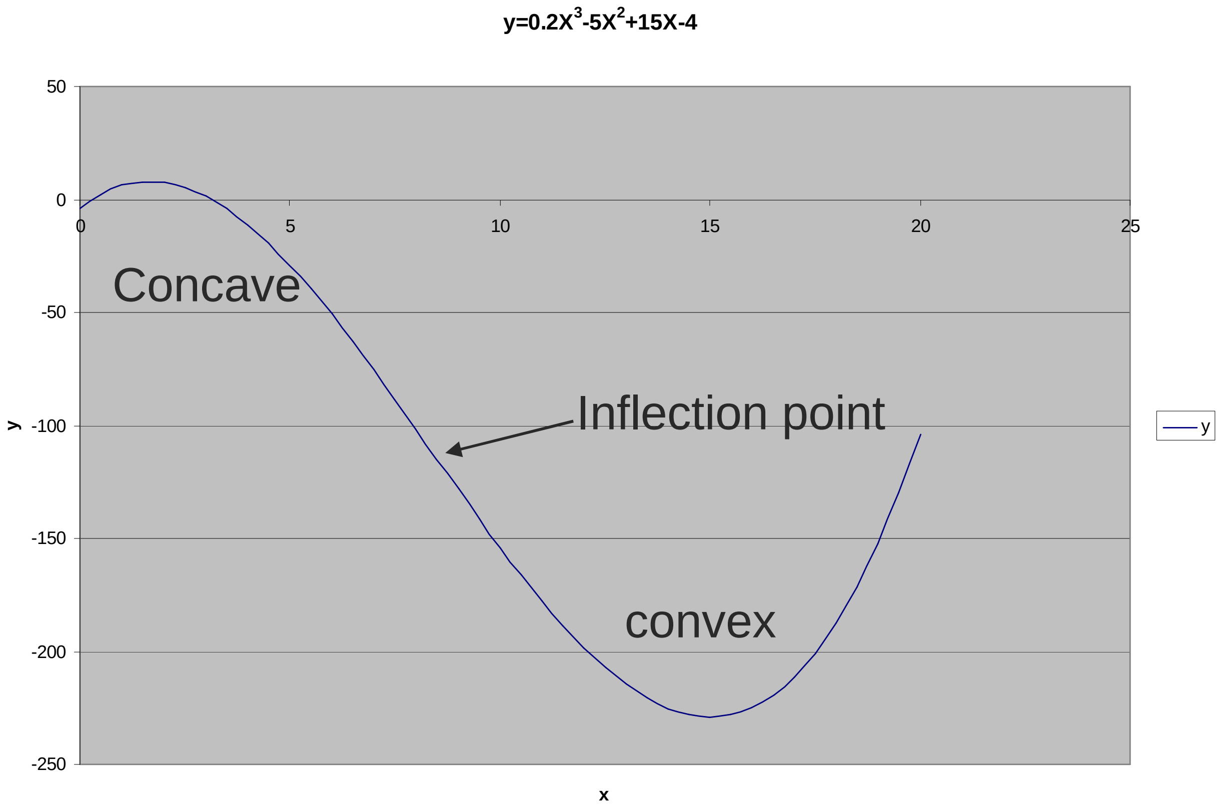
This equals 0 at x = 8.333.

To left of 8.333, the second derivative is negative, hence concave. To the right of 8.333, the second derivative is positive, hence the function is convex in this interval.

Graph of derivatives for example 3



Graph of Example 3



Optimal Excise Tax

Let TR equal total tax revenue,

$TR = (P + T) \cdot Q\text{-}P \cdot Q$ *where*

P = supply Price, T = per unit tax and Q = equilibrium quantity.

To maximize TR, set the derivative of TR with respect to Q equal to zero. Therefore

$$\frac{dTR}{dQ} = \left(Q \cdot \frac{d(P+T)}{dQ} + (P+T) \right) - \left(Q \cdot \frac{dP}{dQ} + P \right) = 0$$

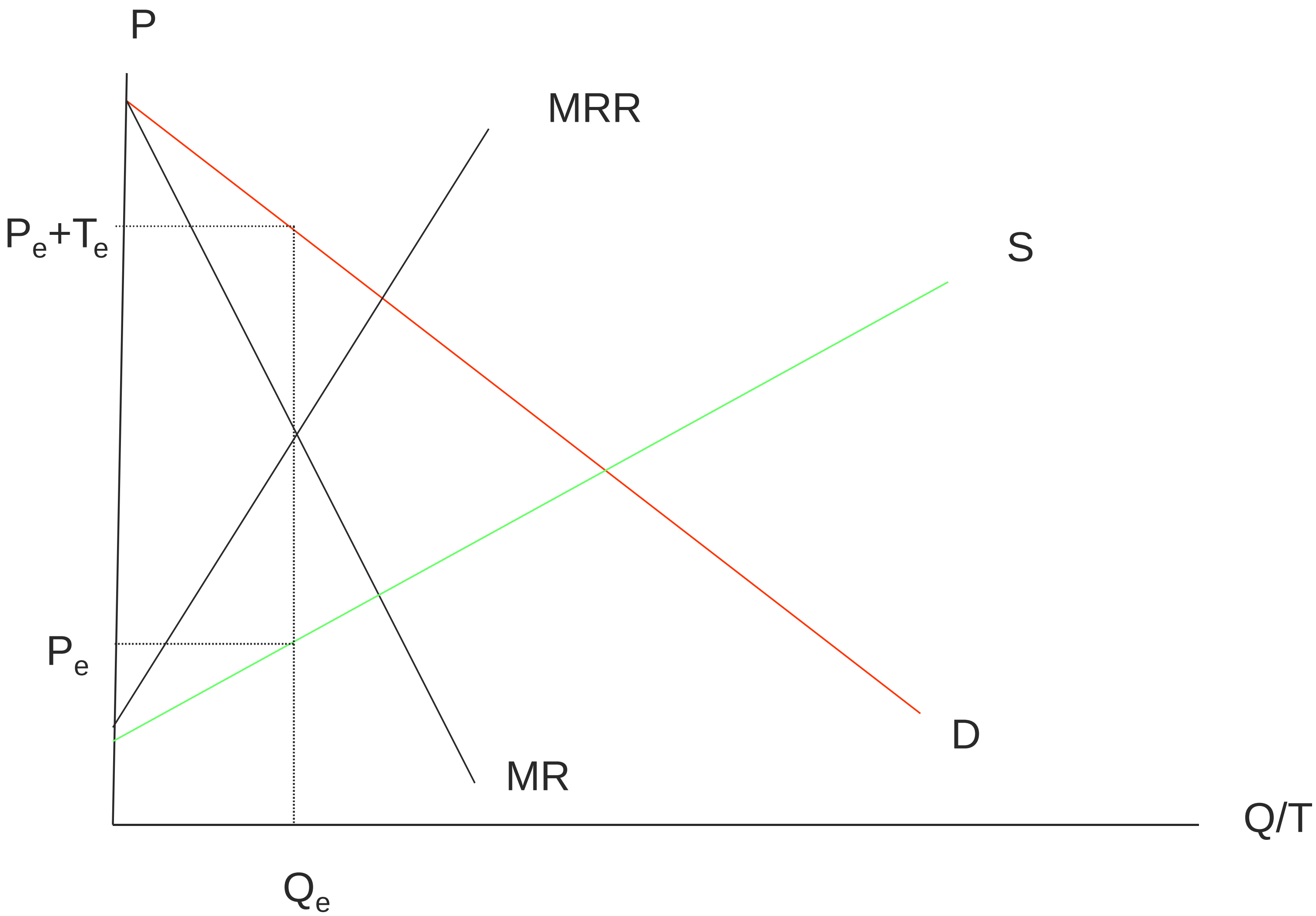
which after careful manipulation reduces to

$$(\mathrm{P} + \mathrm{T}) \cdot \left(1 + \frac{1}{\varepsilon_D} \right) = P \cdot \left(1 + \frac{1}{\varepsilon_S} \right)$$

or $MR = MRR$

MR = marginal revenue and MRR = marginal reservation revenue.

Optimal excise tax rate



Optimal Timing

A wine dealer has wine which value increases with time according to the function $V = Ke^{\sqrt{t}}$. Clearly, the current value is K. Assume that the wine is paid for and there is no storage cost. Maximum profit corresponds to maximum revenue. However, a future dollar is not worth the same as a dollar today. Therefore, we must discount each future revenue to get its present value to make the comparison. We will assume the interest rate on continuous compounding basis is r. Then the present value of V is

$A(t) = Ve^{-rt} = Ke^{\sqrt{t}}e^{-rt} = Ke^{\sqrt{t}-rt}$.

We want to maximize A(t) with respect to t.

Optimal Timing Continued

To maximize our function, we will make use of the fact that a monotonic transformation reach extremes at the same value as the original function. Therefore, we will take the log transformation of our present value function.

$$\ln(\mathrm{A(t)}) = \mathrm{Ln(K)} + \sqrt{\mathrm{t}} - rt$$

When we differentiate both sides, we get

$$\frac{1}{\mathrm{A}}\frac{dA}{dt} = \frac{1}{2}t^{-\frac{1}{2}} - r \quad or \quad \frac{dA}{dt} = A\left(\frac{1}{2}t^{-\frac{1}{2}} - r\right)$$

Which equals zero only when the term in the bracket equal zero since $\mathrm{A} \neq 0$.

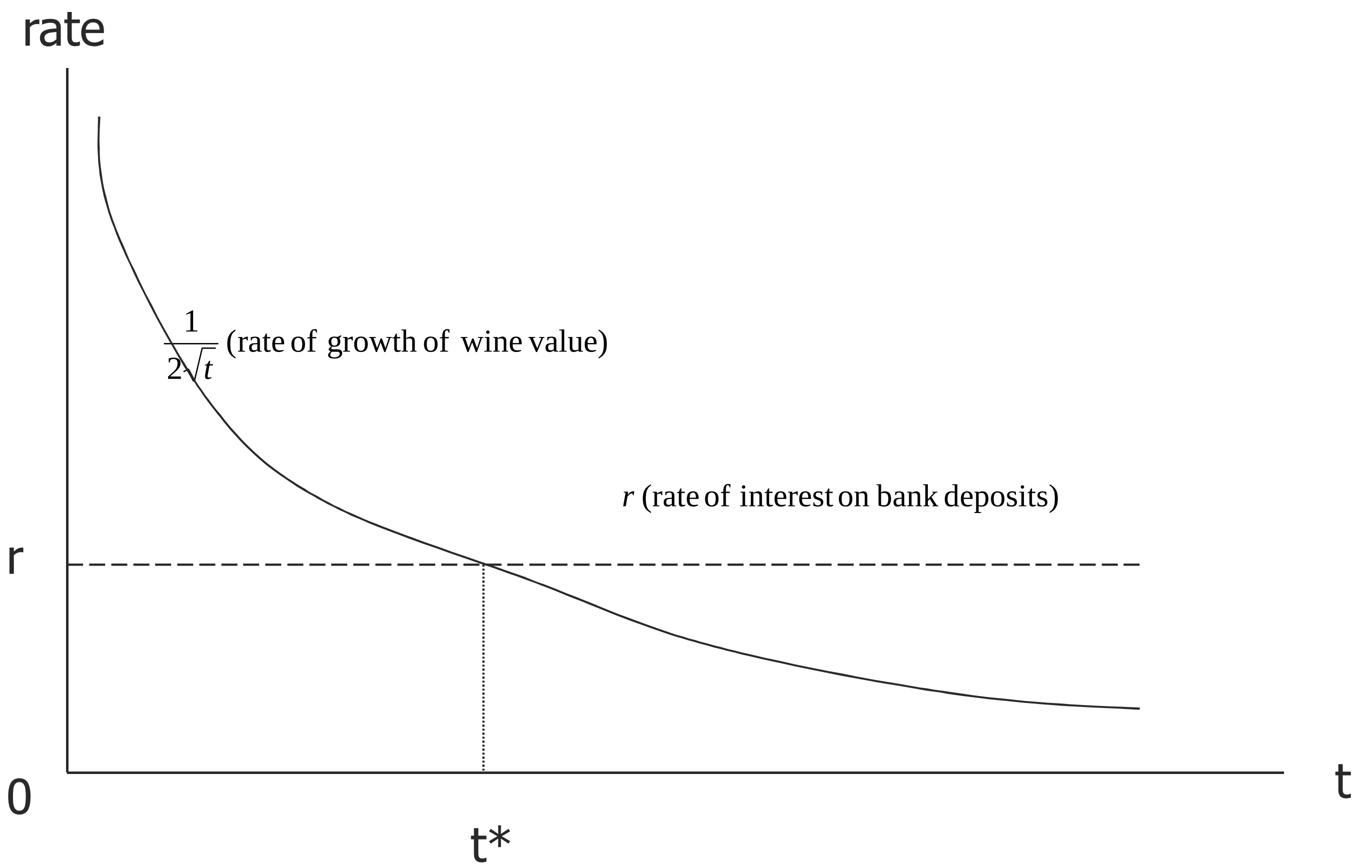
$$\frac{1}{2}t^{-\frac{1}{2}} - r = 0 \Rightarrow t^{-\frac{1}{2}} = 2r \Rightarrow \sqrt{t} = \frac{1}{2r} \Rightarrow t^* = \frac{1}{4r^2}$$

Interpretation of results

Our first order conditions admit an easy interpretation. Note for our value function, the rate of growth of value is $\frac{\mathrm{dln(V)}}{\mathrm{dt}} = \frac{d\left[\ln K + \sqrt{t}\right]}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$.

The rate of growth of money put in the bank with continuous compounding is r (the rate of interest). Thus at our stationary point the rate of growth of value of the wine is equal to the cost of holding money, r.

Interpretation of results



Second Order Condition

$$\frac{d^2A}{dt^2} = \frac{d}{dt} A\left(\frac{1}{2}t^{-\frac{1}{2}} - r\right) = A\frac{d}{dt}\left(\frac{1}{2}t^{-\frac{1}{2}} - r\right) + \left(\frac{1}{2}t^{-\frac{1}{2}} - r\right)\frac{dA}{dt}$$

Since the last term is zero, we have

$$\frac{d^2A}{dt^2} = A\frac{d}{dt}\left(\frac{1}{2}t^{-\frac{1}{2}} - r\right) = A\left(-\frac{1}{4}t^{-\frac{3}{2}}\right) < 0 \text{ at } t^*.$$

Since $A > 0$ at t^*.