Economics 2301

Lecture 32

Multivariate Optimization

Multivariate Optimization

- Many problems in economics reflect a need to choose among alternatives.
- Consumers choose among a wide set of goods.
- Producers choose among input combinations.
- Producers choose among output mixes.
- Policy makers choose among policy tools.

Differential and critical points

The differential *dy* of the univariate function $y=f(x)$ evaluated at the point x_0 is $dy=f'(x_0)dx$. The differential defines a line tangent to the function $f(x)$ at the point x_0. At a stationary point x^*, where $f'(x^*)=0$, the differential *dy* equals zero for any *dx*. Since all interior extreme points of functions that are everywhere differentiable are also stationary points, it follows that the differential equals zero at all interior extreme points of functions that are everywhere differentiable.

Differential and critical points

The multivariate analogue to the differential is the total differential. The total differential dy of the multivariate function $y = f(x_1, x_2, \cdots, x_n)$ is

$dy = f_1(x_1, x_2, \cdots, x_n)dx_1 + \cdots + f_n(x_1, x_2, \cdots, x_n)dx_n$, where

$f_i(x_1, x_2, \cdots, x_n)$ represents the partial derivative of the function with respect to the ith argument. Extending the logic of the univariate case, we say that a stationary point of a multivariate function is a set of values of the arguments of that function $(x_1^*, x_2^*, \cdots, x_n^*)$ such that the total differential dy equals zero for any set of values $(dx_1, dx_2, \cdots, dx_n)$. This occurs at the point where all partial derivaties equal zero.

First-Order Conditions for Multivariate Function

If the function $y = f(x_1, x_2, \cdots, x_n)$ is differentiable with respect to each of its arguments on a domain and reaches a maximum or a minimum at the stationary point $(x_1^*, x_2^*, \cdots, x_n^*)$ within that domain, then each of the partial derivatives evaluated at that point equals zero. That is,

$$
\begin{aligned}
f_1(x_1^*, x_2^*, \cdots, x_n^*) &= 0, \\
f_2(x_1^*, x_2^*, \cdots, x_n^*) &= 0, \\
&\vdots \\
f_n(x_1^*, x_2^*, \cdots, x_n^*) &= 0.
\end{aligned}
$$

Example 1

$$Let \quad y = 10 + 4x_1 - 2x_1^2 - 9x_2 - 1.5x_2^2$$

$$\frac{\partial y}{\partial x_1} = 4 - 4x_1 = 0$$

$$\frac{\partial y}{\partial x_2} = -9 - 3x_2 = 0$$

$\rightarrow x_1^* = 1; x_2^* = -3$ is a critical point. It turns out to be a local maximum.

Example 2

$$Let\ y = 20 - 4x_1^2 + 8x_1x_2 - 2x_2^2 + 16x_1 - 2x_2$$

First Order Conditions :

$$\frac{\partial y}{\partial x_1} = -8x_1 + 8x_2 + 16 = 0$$

$$\frac{\partial y}{\partial x_2} = 8x_1 - 4x_2 - 2 = 0$$

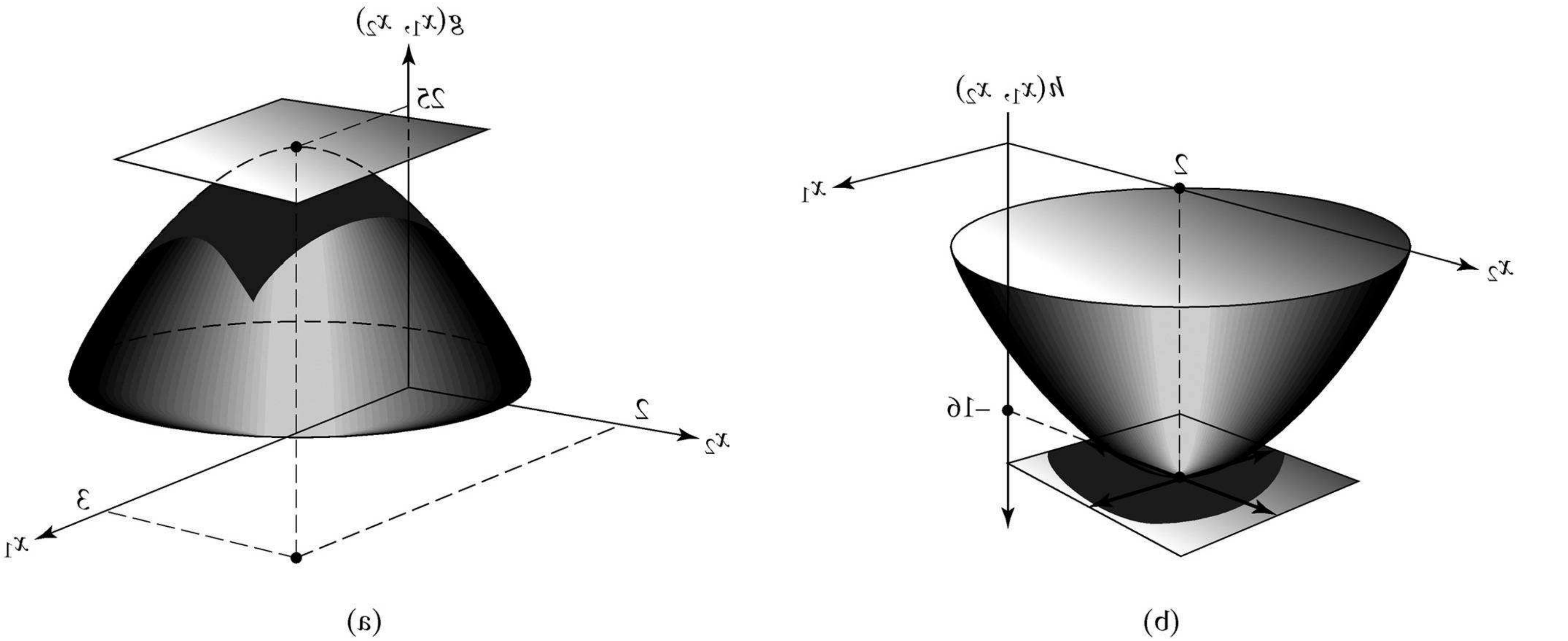
From equation 2, we get $x_2 = 2x_1 - 0.5$. Substituting into 1,

we get $-8x_1 + 8(2x_1 - 0.5) + 16 = 0$ *or*

$$8x_1 + 12 = 0 \Rightarrow x_1^* = -1.5\ and\ x_2^* = -3.5$$

This point turns out to be a saddle point.

Figure 10.1 Stationary Points and the Tangent Planes of Bivariate Functions



Economic Example – Multiproduct Firm

We have a multiproduct firm working in competive markets.
They can sell all they want of each product at prices P_1 and P_2. The firms revenue function is $R = P_1Q_1 + P_2Q_2$.
The firm's cost function is assumed to be
$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$. Note that marginal cost of Q_1 depends on level of Q_2. $\frac{\partial C}{\partial Q_1} = 4Q_1 + Q_2$. Our profit function is

$$\pi = R - C = P_1Q_1 + P_2Q_2 - 2Q_1^2 - Q_1Q_2 - 2Q_2^2$$

Multiproduct Example Cont.

$$\pi = \mathrm{R - C} = P_1 Q_1 + P_2 Q_2 - 2\mathrm{Q}_1^2 - Q_1 Q_2 - 2Q_2^2$$

First Order conditions for profit maximization :

$$\pi_1 = P_1 - 4Q_1 - Q_2 = 0$$

$$\pi_2 = P_2 - Q_1 - 4Q_2 = 0$$

Our optimal outputs are

$$Q_1^* = \frac{4P_1 - P_2}{15} \quad and \quad Q_2^* = \frac{4P_2 - P_1}{15}$$

Multiproduct Monopolist

Our monopolist faces the following demand functions for the two products :

$$Q_1 = 40 - 2P_1 + P_2$$

$$Q_2 = 15 + P_1 - P_2$$

Converting these to average revenue functions

$$P_1 = 55 - Q_1 - Q_2$$

$$P_2 = 70 - Q_1 - 2Q_2$$

Our total revenue function is now

$$R = P_1Q_1 + P_2Q_2 = (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2$$

$$= 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2$$

Multiproduct Monopolist Cont.

$$\text{Revenue Function}: R = 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2$$

$$\text{Total Cost}: \mathrm{C} = Q_1^2 + Q_1Q_2 + Q_2^2$$

$$\text{Profit}: \pi = R - C = 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2 - \left(Q_1^2 + Q_1Q_2 + Q_2^2\right)$$

$$= 55Q_1 - 2Q_1^2 + 70Q_2 - 3Q_2^2 - 3Q_1Q_2$$

First Order Conditions:

$$\pi_1 = 55 - 4Q_1 - 3Q_2 = 0$$

$$\pi_2 = 70 - 3Q_1 - 6Q_2 = 0$$

This gives us

$$\mathrm{Q}_1^* = 8 \quad and \quad Q_2^* = 7\frac{2}{3}$$

Price Discrimination

We have a monopolist who sells in two markets, the firm's total revenue is $R = R_1(Q_1) + R_2(Q_2)$ and total cost is $C = C(Q)$ where $Q = Q_1 + Q_2$.

$$\pi = R - C = R_1(Q_1) + R_2(Q_2) - C(Q)$$

First Order Conditions :

$$\pi_1 = R_1^{'}(Q_1) - C'(Q)\frac{\partial Q}{\partial Q_1} = R_1^{'}(Q_1) - C'(Q) = 0$$

$$\pi_1 = R_2^{'}(Q_2) - C'(Q)\frac{\partial Q}{\partial Q_2} = R_2^{'}(Q_2) - C'(Q) = 0$$

That is $MC = MR_1 = MR_2$.

Price Discrimination Cont.

$$\text{Now } \mathrm{R_i} = P_i Q_i \quad i = 1,2$$

$$MR_i = \frac{\partial R_i}{\partial Q_i} = P_i \frac{dQ_i}{dQ_i} + Q_i \frac{dP_i}{dQ_i} = P_i\left(1 + \frac{Q_i}{P_i}\frac{dP_i}{dQ_i}\right)$$

$$= P_i\left(1 + \frac{1}{\varepsilon_{d_i}}\right) = P_i\left(1 - \frac{1}{\left|\varepsilon_{d_i}\right|}\right)$$

Note that marginal revenue is positive only if demand is price elastic. For our monopolist marginal revenue is same in both markets and equals marginal cost, therefore monopolist charges higher price in market with lower elasticity.