Economics 2301

Lecture 33

Multivariate Optimization

Multiproduct Monopolist

Our monopolist faces the following demand functions for the two products :

$$Q_1 = 40 - 2P_1 + P_2$$

$$Q_2 = 15 + P_1 - P_2$$

Converting these to average revenue functions

$$P_1 = 55 - Q_1 - Q_2$$

$$P_2 = 70 - Q_1 - 2Q_2$$

Our total revenue function is now

$$R = P_1Q_1 + P_2Q_2 = (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2$$

$$= 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2$$

Multiproduct Monopolist Cont.

$$\text{Revenue Function}: R = 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2$$

$$\text{Total Cost}: \mathrm{C} = Q_1^2 + Q_1Q_2 + Q_2^2$$

$$\text{Profit}: \pi = R - C = 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2 - \left(Q_1^2 + Q_1Q_2 + Q_2^2\right)$$

$$= 55Q_1 - 2Q_1^2 + 70Q_2 - 3Q_2^2 - 3Q_1Q_2$$

First Order Conditions:

$$\pi_1 = 55 - 4Q_1 - 3Q_2 = 0$$

$$\pi_2 = 70 - 3Q_1 - 6Q_2 = 0$$

This gives us

$$\mathrm{Q}_1^* = 8 \quad \textit{and} \quad Q_2^* = 7\frac{2}{3}$$

Price Discrimination

We have a monopolist who sells in two markets, the firm's total revenue is $R = R_1(Q_1) + R_2(Q_2)$ and total cost is $C = C(Q)$ where $Q = Q_1 + Q_2$.

$$\pi = R - C = R_1(Q_1) + R_2(Q_2) - C(Q)$$

First Order Conditions:

$$\pi_1 = R_1^{'}(Q_1) - C'(Q)\frac{\partial Q}{\partial Q_1} = R_1^{'}(Q_1) - C'(Q) = 0$$

$$\pi_1 = R_2^{'}(Q_2) - C'(Q)\frac{\partial Q}{\partial Q_2} = R_2^{'}(Q_2) - C'(Q) = 0$$

That is $MC = MR_1 = MR_2$.

Price Discrimination Cont.

$$\text{Now } R_i = P_i Q_i \quad i = 1,2$$

$$MR_i = \frac{\partial R_i}{\partial Q_i} = P_i \frac{dQ_i}{dQ_i} + Q_i \frac{dP_i}{dQ_i} = P_i \left(1 + \frac{Q_i}{P_i} \frac{dP_i}{dQ_i} \right)$$

$$= P_i \left(1 + \frac{1}{\varepsilon_{d_i}} \right) = P_i \left(1 - \frac{1}{\left| \varepsilon_{d_i} \right|} \right)$$

Note that marginal revenue is positive only if demand is price elastic. For our monopolist marginal revenue is same in both markets and equals marginal cost, therefore monopolist charges higher price in market with lower elasticity.

Second Order Conditions

Second Differential of a function can be considered as the differential of the first differential and therefore is denoted as $d(dy) = d^2y$. The second differential of the univariate function $y = f(x)$ equals

$$d^2y = \frac{d\left(f'(x)dx\right)}{dx}dx = \left(f''(x)dx\right)dx = f''(x)(dx)^2$$

which is the product of the second derivative and the term $(dx)^2$, which is positive for an dx.

Second Order Conditions

- A sufficient condition for a stationary point of a univariate function to be a local maximum is that the second derivative evaluated at that point is negative. This implies that a sufficient condition for a local maximum is that the second differential at the point be negative.
- A sufficient condition for a local minimum of a univariate function is that the second differential evaluated at that point is positive for any *dx*.