Economics 2301

Lecture 34

Multivariate Optimization

Second Total Differential of Bivariate Function

For the bivariate function $y = f(x_1, x_2)$, our total differential is

$dy = f_1(x_1, x_2)dx_1 + f_2(x_1, x_2)dx_2$.

Taking the total derivative of the total differential by treating the dx_i terms as constants and f_i terms as functions, we get the second total differential

$$d^2y = \frac{\partial\left(f_1 dx_1 + f_2 dx_2\right)}{\partial x_1} dx_1 + \frac{\partial\left(f_1 dx_1 + f_2 dx_2\right)}{\partial x_2} dx_2$$

$$= f_{11}\left(dx_1\right)^2 + f_{22}\left(dx_2\right)^2 + 2 f_{12} dx_1 dx_2$$

Example of Second Total Differential

$$Let\ y = \ln(x_1)\ln(x_2)$$

The total differential is

$$dy = \frac{\ln(x_2)}{x_1}dx_1 + \frac{\ln(x_1)}{x_2}dx_2$$

The second total differential is

$$\mathrm{d}^2 y = -\frac{\ln(x_2)}{x_1^2}(dx_1)^2 - \frac{\ln(x_1)}{x_2^2}(dx_2)^2 + \frac{2}{x_1 x_2}dx_1 dx_2$$

Second Order Conditions

- If the second total differential evaluated at a stationary point of a function $f(x_1,x_2)$ is negative for any dx_1 and dx_2, then that stationary point represents a local maximum of the function.
- If the second total differential evaluated at a stationary point of a function $f(x_1,x_2)$ is positive for any dx_1 and dx_2, then that stationary point represents a local minimum of the function.

Deriving the second order conditions

Completing the square of our second total differential by adding and subtracting $(f_{12})^2(dx_2)^2 / f_{11}$, we get

$$d^2y = f_{11}\left[(dx_1)^2 + 2\frac{f_{12}}{f_{11}}dx_1dx_2 + \left(\frac{f_{12}}{f_{11}}\right)^2(dx_2)^2\right]$$

$$+\left[f_{22} - \frac{(f_{12})^2}{f_{11}}\right](dx_2)^2$$

$$= f_{11}\left(dx_1 + \frac{f_{12}}{f_{11}}dx_2\right)^2 + \left[f_{22} - \frac{(f_{12})^2}{f_{11}}\right](dx_2)^2$$

Deriving the second order conditions

The second total differential is positive for any values of dx_1 and dx_2 if f_{11} is positive and if

$$f_{22} - \frac{(f_{12})^2}{f_{11}} > 0$$ or, equivalently, if f_{11} is positive and if

$$f_{11} f_{22} > (f_{12})^2.$$

The second total differential is negative for any values of dx_1 and dx_2 if f_{11} is negative and if

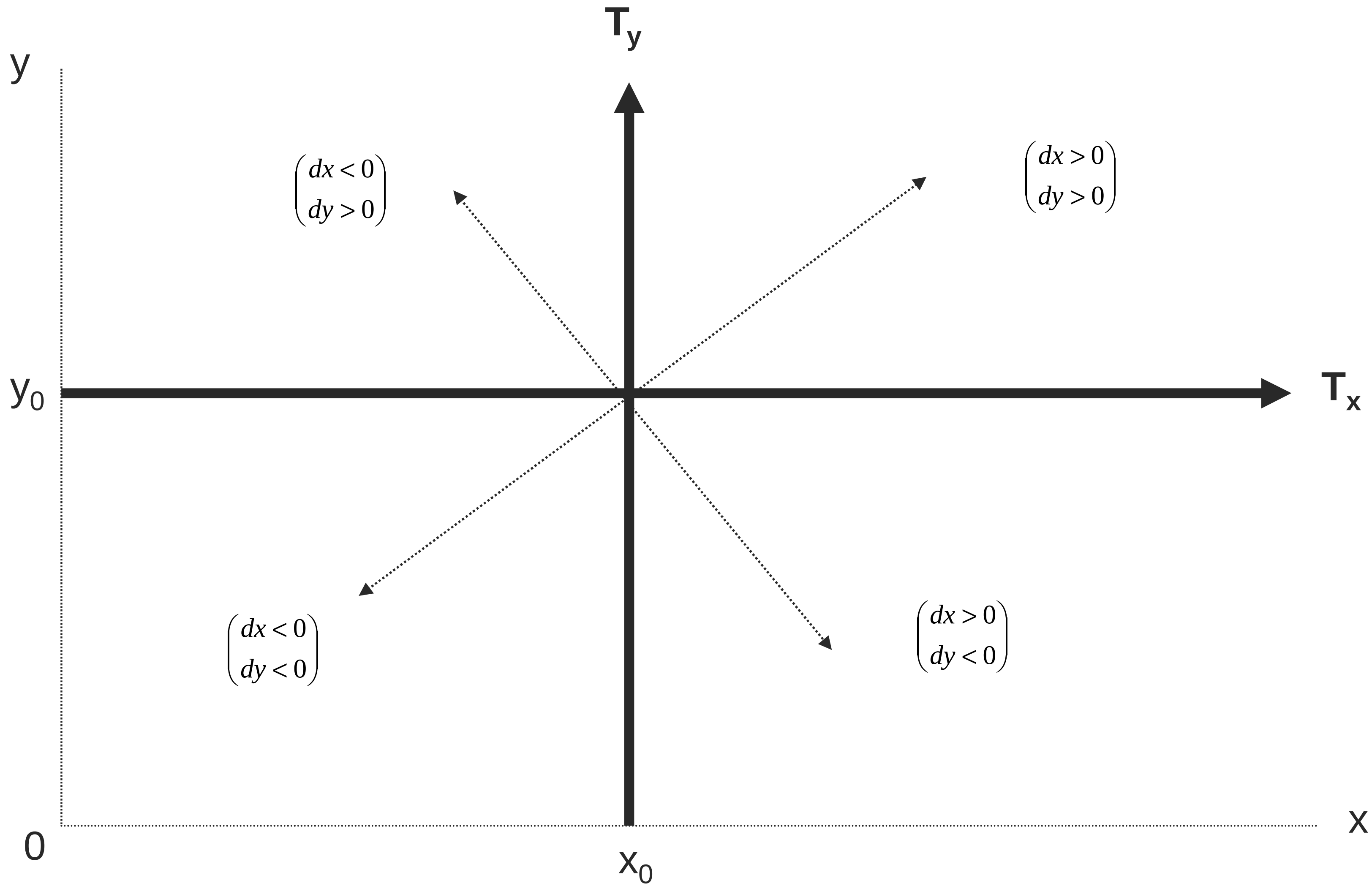
$$f_{22} - \frac{(f_{12})^2}{f_{11}} < 0$$ or, equivalently, if f_{11} is negative and if

$$f_{11} f_{22} > (f_{12})^2.$$

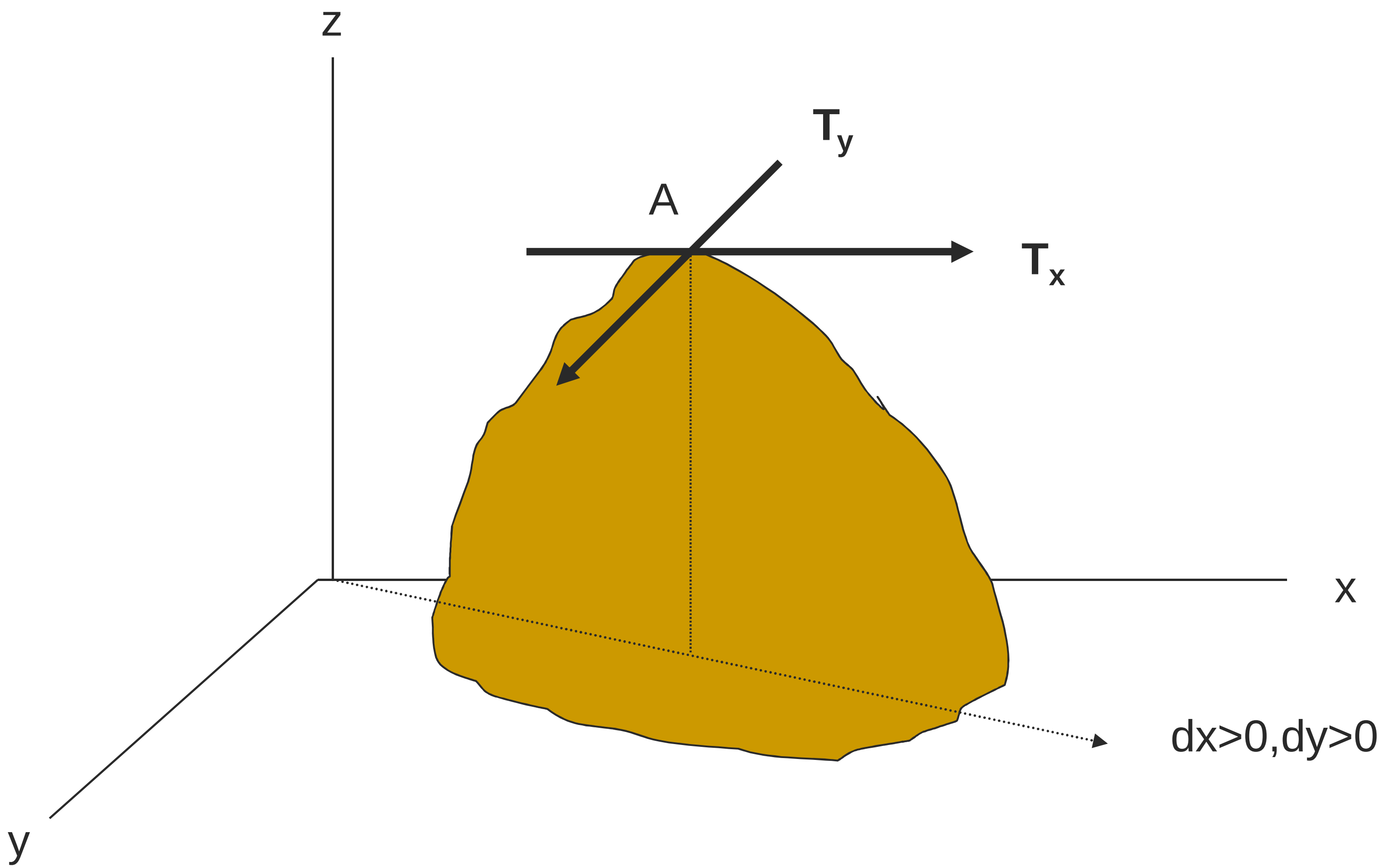
Understanding the 2nd Order condition

For the function, $z = f(x,y)$, the sign of d^2z hinges not only on f_{xx} and f_{yy}, which have to do with the surface configurat ion arond point A (figure next slide) in the two basic directions shown by T_x (east - west) and T_y (north - south), but also on the cross partial derivative f_{xy}. The role played by this latter partial derivative is to ensure that the surface in question w ill yield (two - dimensiona l) cross sections with the same type of configurat ion (hill or valley, as the case may be) not only in the two basic directions (east - west and north - south), but in all other possible directions (such as northeast - southwest) as well.

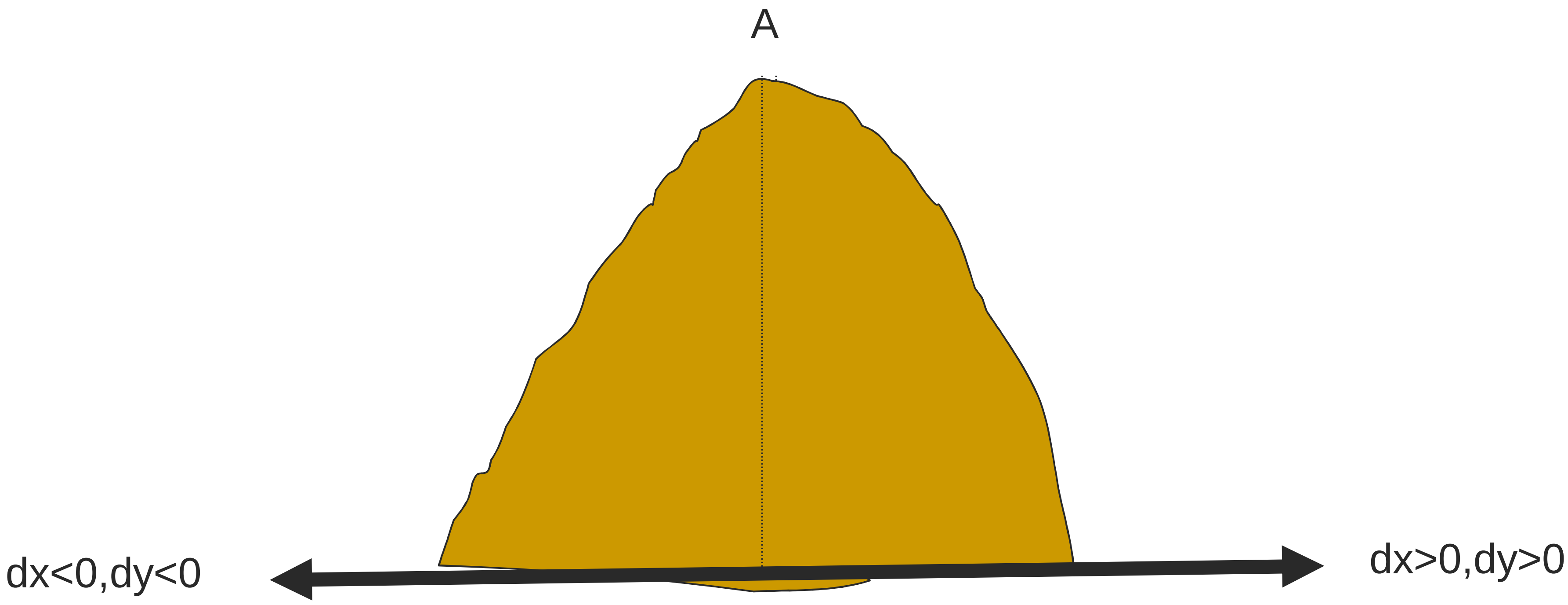
Understanding the 2nd Order Conditions



Maximum drawing



Our Slice of the cone



2^{nd} Order conditions for Bivariate Function

If the function $y = f(x_1, x_2)$ has the stationary point (x_1^*, x_2^*)

and if $f_{11}(x_1^*, x_2^*) < 0$ and $f_{11}(x_1^*, x_2^*) f_{22}(x_1^*, x_2^*) > \left(f_{12}(x_1^*, x_2^*) \right)^2$

then the function reaches a local maximum at this stationary point.

If the function $y = f(x_1, x_2)$ has the stationary point (x_1^*, x_2^*)

and if $f_{11}(x_1^*, x_2^*) > 0$ and $f_{11}(x_1^*, x_2^*) f_{22}(x_1^*, x_2^*) > \left(f_{12}(x_1^*, x_2^*) \right)^2$

then the function reaches a local minimum at this stationary point.

Example

$$Let \quad f(x_1, x_2) = 100 + 5x_1 - 4x_2 + 2.5x_1^2 + 6x_2^2 - 4x_1x_2$$

First Order Conditions :

$$f_1 = 5 + 5x_1 - 4x_2 = 0$$

$$f_2 = -4 - 4x_1 + 12x_2 = 0$$

$$\Rightarrow x_1^* = -1, \ x_2^* = 0$$

2nd Order Conditions

$$f_{11} = 5 > 0, \ f_{22} = 12 > 0, \ f_{12} = -4$$

$$f_{11} f_{22} = 5 \cdot 12 = 60 > 16 = (-4)^2 = (f_{12})^2$$

$\therefore$ We have a minimum.

Multiproduct Monopolist

Our monopolist faces the following demand functions

for the two products:

$Q_1 = 40 - 2P_1 + P_2$

$Q_2 = 15 + P_1 - P_2$

Converting these to average revenue functions

$P_1 = 55 - Q_1 - Q_2$

$P_2 = 70 - Q_1 - 2Q_2$

Our total revenue function is now

$R = P_1Q_1 + P_2Q_2 = (55 - Q_1 - Q_2)Q_1 + (70 - Q_1 - 2Q_2)Q_2$

$= 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2$

Multiproduct Monopolist Cont.

$$\text{Revenue Function}: R = 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2$$

$$\text{Total Cost}: \text{C} = Q_1^2 + Q_1Q_2 + Q_2^2$$

$$\text{Profit}: \pi = R - C = 55Q_1 - Q_1^2 + 70Q_2 - 2Q_2^2 - 2Q_1Q_2 - \left(Q_1^2 + Q_1Q_2 + Q_2^2\right)$$

$$= 55Q_1 - 2Q_1^2 + 70Q_2 - 3Q_2^2 - 3Q_1Q_2$$

First Order Conditions:

$$\pi_1 = 55 - 4Q_1 - 3Q_2 = 0$$

$$\pi_2 = 70 - 3Q_1 - 6Q_2 = 0$$

This gives us

$$\text{Q}_1^* = 8 \quad \textit{and} \quad Q_2^* = 7\frac{2}{3}$$

Multiproduct Monopolist Cont.

First Order Conditions :

$$\pi_1 = 55 - 4Q_1 - 3Q_2 = 0$$

$$\pi_2 = 70 - 3Q_1 - 6Q_2 = 0$$

Second Order Conditions :

$$\pi_{11} = -4 < 0, \pi_{22} = -6 < 0, \pi_{12} = -3$$

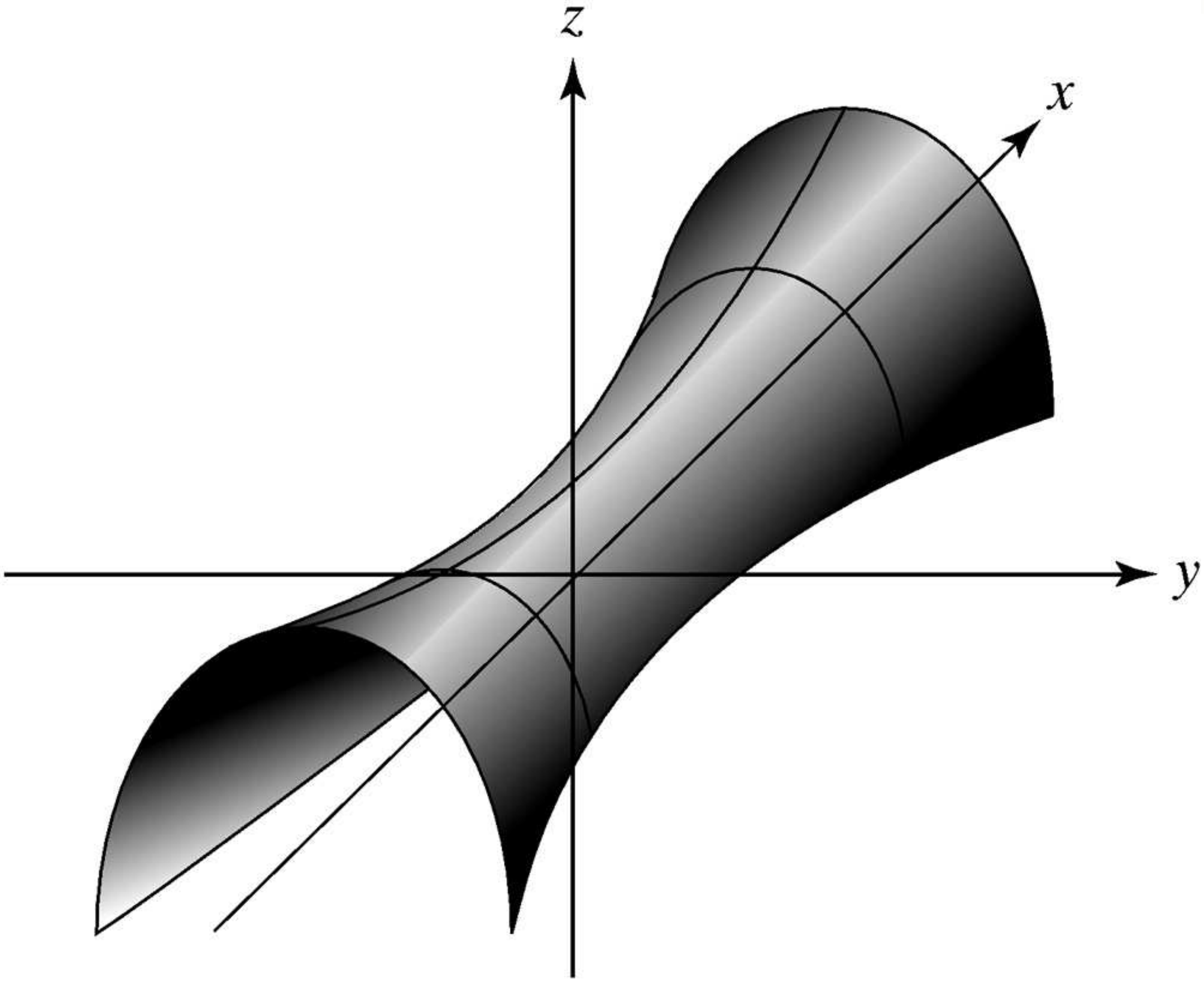
$$\pi_{11}\pi_{22} = (-4)(-6) = 24 > 9 = (-3)^2 = (\pi_{12})^2$$

$\therefore$ Profits are maximized.

Saddle Point

One possibility not covered by these sufficient conditions is that one 2nd-order partial derivative is positive and the other is negative, for example, $f_{11} > 0$ and $f_{22} < 0$. In this case the stationary point is called a saddle point, so named because the function looks like a saddle.

Figure 10.4 A Saddle Point



Example of Saddle Point

$$Let \quad f(x_1, x_2) = 100 + 5x_1 - 4x_2 + 2.5x_1^2 - 6x_2^2 - 4x_1x_2$$

First Order Conditions:

$$f_1 = 5 + 5x_1 - 4x_2 = 0$$

$$f_2 = -4 - 4x_1 - 12x_2 = 0$$

$$\Rightarrow x_1^* = -1, \; x_2^* = 0$$

2nd Order Conditions

$$f_{11} = 5 > 0, \; f_{22} = 12 < 0, \; f_{12} = -4$$

$$f_{11} f_{22} = 5 \cdot (-12) = -60 < 16 = (-4)^2 = (f_{12})^2$$

$\therefore$ We have a saddlepoint.

Example 2 from last Lecture

$$Let\ \ y = 20 - 4x_1^2 + 8x_1x_2 - 2x_2^2 + 16x_1 - 2x_2$$

First Order Conditions :

$$\frac{\partial y}{\partial x_1} = -8x_1 + 8x_2 + 16 = 0$$

$$\frac{\partial y}{\partial x_2} = 8x_1 - 4x_2 - 2 = 0$$

From equation 2, we get $x_2 = 2x_1 - 0.5$. Substituting into 1,

we get $-8x_1 + 8(2x_1 - 0.5) + 16 = 0$ *or*

$$8x_1 + 12 = 0 \Rightarrow x_1^* = -1.5\ and\ \ x_2^* = -3.5$$

Example 2 Continued

$$f_{11} = -8 < 0,\ f_{22} = -4 < 0,\ f_{12} = 8$$

$$f_{11} f_{22} = (-8)(-4) = 32 < 64 = 8^2 = (f_{12})^2$$

2nd Order conditions not satisfied.

We are not sure what we have. It

may be saddlepoint or inflection point.

Graph of Previous Slide

- http://www.compute.uwlax.edu/calc2D/output/2112/