Economics 2301

Lecture 35
Multivariate Optimization III
Second differential as Quadratic form

\[ d^2 y = f_{11} \, dx_1^2 + 2f_{12} \, dx_1 \, dx_2 + f_{22} \, dx_2^2 = \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \]

This makes the second total differential and quadratic form in \( dx_1 \) and \( dx_2 \). In general a multivariate quadratic form can be written as:

\[ Q = a_{11} \, z_1^2 + 2a_{12} \, z_1 \, z_2 + \cdots + 2 \, a_{ij} \, z_i \, z_j + \cdots + a_{nn} \, z_n^2 = z' \, Az \]

Continued
Second Differential as Quadratic Form

\[ Q = z' A z = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \]

A is a symmetric matrix. In our problem the \( z \) variables are the differentials \( dx_i \) and the matrix A is composed of the second partial derivatives of \( y, f_{ij}, \) where \( i, j = 1, 2, \ldots, n \). (see next slide).
Second Order Total Differential

\[ d^2 y = dx' H dx = \begin{bmatrix} dx_1 & dx_2 & \cdots & dx_n \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} \]

The matrix of second partial derivatives is called the Hessian matrix and is symmetric because of ?? Theorem.
A quadratic form $z'Az$ is **positive definite** if, for any column vector $z$ consisting of the $n$ elements $z_i$, $i = 1,2,...,n$, other than the zero vector, the quadratic form is always positive. A quadratic form $z'Az$ is **negative definite** if, for any column vector $z$ consisting of the $n$ elements $z_i$, $i = 1,2,...,n$, other than the zero vector, the quadratic form is always negative. Necessary and sufficient conditions for determining whether a matrix is positive or negative definite concern the sign of its leading principal minors. The leading principal minors of a matrix are the determinants of its leading principal submatrices. The $k$th leading principal submatrix of any $nXn$ matrix is the $kXk$ matrix obtained by deleting the last $n-k$ rows and the last $n-k$ column of the matrix. An $nXn$ matrix has $n$ leading principal minors.
Example

Let $A = \begin{bmatrix}
a & b & c & d \\
e & f & g & h \\
i & j & k & l \\
m & n & o & p
\end{bmatrix}$

*first principal submatrix is $[a]$, the first principal minor is $a$.*

*second principal submatrix is $\begin{bmatrix}a & b \\e & f\end{bmatrix}$, 2nd principal minor is $af - be$.***
The third principal submatrix is

\[
\begin{bmatrix}
  a & b & c \\
  e & f & g \\
  i & j & k
\end{bmatrix}
\]

The third principal minor is \( afk + ejc + igb - cfj - gja - keb \)

The fourth principal submatrix is \( A \) itself; its principal minor is the determinant of \( A \).
Definiteness

- An nXn matrix is negative definite if and only if all of its n leading principal minors alternate in sign with the first principal minor negative.
- An nXn matrix is positive definite if and only if all of its n leading principal minors are strictly positive.
Extremeums

- If a multivariate function has a stationary point and the Hessian of this function evaluated at that stationary point is negative definite, then this stationary point represents a local maximum of the function.
- If a multivariate function has a stationary point and the Hessian of this function evaluated at that stationary point is positive, then this stationary point represents a local minimum of the function.
Numerical Example

\[ y = 2x_1x_2 - \frac{1}{2}x_1^2 - 3x_2^2 + x_2x_3 - 1.5x_3^2 = 10x_3 \]

\[ \frac{\partial y}{\partial x_1} = -x_1 + 2x_2 = 0 \]

\[ \frac{\partial y}{\partial x_2} = 2x_1 - 6x_2 + x_3 = 0 \]

\[ \frac{\partial y}{\partial x_3} = x_2 - 3x_3 + 10 = 0 \]

*By substitution* \( x_1 = 10, x_2 = 5, x_3 = 10 *
Example Continued

$$H = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -6 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

first principal minor $= -1$

second principal minor $= \begin{vmatrix} -1 & 2 \\ 2 & -6 \end{vmatrix} = 6 - 4 = 2$

$|H| = -18 + 1 + 12 = -5$

We have a maximum