Economics 2301

Lecture 35

Multivariate Optimization III

Second differential as Quadratic form

$$d^2 y = f_{11} dx_1^2 + 2\mathrm{f}_{12} dx_1 dx_2 + f_{22} dx_2^2 = \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix}$$

This makes the second total differential and quadratic form in dx_1 and dx_1. In general a multivariate quadratic form can be written as:

$$Q = a_{11} z_1^2 + 2\mathrm{a}_{12} z_1 z_2 + \cdots + 2\, a_{ij} z_i z_j + \cdots + a_{nn} z_n^2 = z' Az$$

Continued

Second Differential as Quadratic Form

$$Q = z' A z = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

A is a symmetric matrix. In our problem the z_is are the differentials dx_i and the matrix A is composed of the second partial derivatives of y, f_{ij}, i,j = 1, 2, ... , n. (see next slide).

Second Order Total Differential

$$d^2y = dx'H\,dx = \begin{bmatrix} dx_1 & dx_2 & \cdots & dx_n \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$$

H the matrix of second partial derivatives is called the Hessian matrix and is symmetric because of ?? Theorem.

Definiteness of matrix

A quadratic form *z′Az* is **positive definite** if, for any column vector *z* consisting of the n elements z_i, i = 1,2,...,n, other than the zero vector, the quadratic form is always positive. A quadratic form *z′Az* is **negative definite** if, for any column vector *z* consisting of the n elements z_i, i = 1,2,...,n, other than the zero vector, the quadratic form is always negative. Necessary and sufficient conditions for determining whether a matrix is positive or negative definite concern the sign of its leading principal minors. The leading principal minors of a matrix are the determinants of its leading principal submatrices. The *k*th leading principal submatrix of any *nXn* matrix is the *kXk* matrix obtained by deleting the last *n-k* rows and the last *n-k* column of the matrix. An *nXn* matrix has *n* leading principal minors.

Example

$$Let\ A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$first\ principal\ submatrix\ is\ [a],\ the\ first\ principal\ minor\ is\ a.$$

$$second\ principal\ submatrix\ is \begin{bmatrix} a & b \\ e & f \end{bmatrix},\ \text{2nd}\ principal\ minor\ is\ af - be.$$

Example Continued

$$\text{The third principal submatrix is}$$

$$\begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix}$$

$$\text{The third principal minor is } afk + ejc + igb - cfi - gja - keb$$

$$\text{The fourth principal submatrix is } A \text{ itself };$$

$$\text{its principal minor is the determinant of } A.$$

Definiteness

- An nXn matrix is negative definite if and only if all of its n leading principal minors alternate in sign with the first principal minor negative.
- An nXn matrix is positive definite if and only if all of its n leading principal minors are strictly positive.

Extremeums

- If a multivariate function has a stationary point and the Hessian of this function evaluated at that stationary point is negative definite, then this stationary point represents a local maximum of the function.
- If a multivariate function has a stationary point and the Hessian of this function evaluated at that stationary point is positive, then this stationary point represents a local minimum of the function.

Numerical Example

$$y = 2x_1x_2 - \frac{1}{2}x_1^2 - 3x_2^2 + x_2x_3 - 1.5x_3^2 = 10x_3$$

$$\partial y / \partial x_1 = -x_1 + 2x_2 = 0$$

$$\partial y / \partial x_2 = 2x_1 - 6x_2 + x_3 = 0$$

$$\partial y / \partial x_3 = x_2 - 3x_3 + 10 = 0$$

$$By\ substitution\ x_1 = 10, x_2 = 5, x_3 = 10$$

Example Continued

$$H = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -6 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$first\ principal\ minor = -1$$

$$second\ principal\ minor = \begin{vmatrix} -1 & 2 \\ 2 & -6 \end{vmatrix} = 6 - 4 = 2$$

$$|H| = -18 + 1 + 12 = -5$$

$$We\ have\ a\ maximum$$