Economics 2301

Lecture 36

Constrained Optimization
Constrained Optimization

- Much of economics analysis models choice in the face of trade-offs in one form or another.
- Constrained optimization problems appear in virtually every area of economics.
- The basic constrained optimization problem reflects a tension between what is desired and what is obtainable.
- The framework of any constrained optimization problem includes an objective function and one or more constraints.
Constrained Optimization

- The **feasible set** of a constrained optimization problem is the set of arguments of the objective function that satisfies all the constraints of the problem simultaneously.
- **Equality constraints** are constraints that hold exactly.
- **Inequality constraints** allow a function of one or more of the choice variables to be less than or greater than some level.
Solving Constrained Optimization Problems Through Substitution

- Constraints limit the feasible set of choices in an optimization problem.
- One way to solve a constrained optimization problem is to convert it to an unconstrained optimization problem by “internalizing” the constraint(s) directly into the objective function.
- The constraint is internalized when we express it as a function of one of the arguments of the objective function and then substitute out that argument using the constraint.
Find the maximum of the bivariate function

\[ f(x_1, x_2) = 2\sqrt{x_1} + \frac{1}{2} \sqrt{x_2} \]

subject to the constraint: \( 4x_1 + x_2 = 20 \)

Now: \( x_2 = 20 - 4x_1 \)

The internalized objective function is

\[ \tilde{f}(x_1) = 2\sqrt{x_1} + \frac{1}{2} \sqrt{20 - 4x_1} \]

First Order Condition: \( \tilde{f}'(x_1) = \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{20 - 4x_1}} = 0 \)
First Order Condition: \( f'(x_1) = \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{20 - 4x_1}} = 0 \)

This equals zero \( x_1 = 20 - 4x_1 \) or \( 20 - 5x_1 = 0 \)

\( \Rightarrow x_1^* = 4 \) and \( x_2^* = 4 \)

Second Order Condition:

\[ f''(x_1) = -\frac{1}{2} x_1^{-3/2} - 2 \cdot (20 - 4x_1)^{-3/2} \]

at \( x_1 = 4; \quad f''(x_1) = -\frac{1}{2} \cdot \frac{1}{8} - 2 \cdot \frac{1}{8} = -\frac{5}{16} < 0 \)

\( \therefore \) The internalized function is maximized.
Figure 11.1 Constrained Optimization and an Internalized Function
Example 2

Let \( z = 20x - 2x^2 + 30y - 3y^2 + 4xy \)

Unconstrained Optimum:

1st Order conditions:
\( z_x = 20 - 4x + 4y = 0 \)
\( z_y = 30 + 4x - 6y = 0 \)
\( x^* = 30, \ y^* = 25, \ z^* = 675 \)

2nd Order Conditions: \( z_{xx} = -4 < 0, \ z_{yy} = -6 < 0, \ z_{xy} = 4 \)
\( z_{xx}z_{yy} = (-4)(-6) = 24 > 16 = 4^2 = z_{xy}^2 \)
\[ \therefore \text{a maximum.} \]
Example 2 Continued

Now consider maximizing our function subject to the constraint: \( x + y = 40 \implies y = 40 - x \)

Our internalized objective function is

\[
\overline{z} = 20x - 2x^2 + 30(40 - x) - 3(40 - x)^2 + 4x(40 - x)
\]

\[
= 20x - 2x^2 + 1200 - 30x - 3(1600 - 80x + x^2) + 160x - 4x^2
\]

\[
= -3600 + 390x - 9x^2
\]

First Order Condition: \( \overline{z}_x = 390 - 18x = 0 \implies x^* = 21 \frac{2}{3}, \; y^* = 18 \frac{1}{3} \)

Second Order Condition: \( \overline{z}_{xx} = -18 < 0, \therefore \text{maximum.} \)

\( \overline{z} = 625 \)
Utility maximization Problem

Suppose peanuts sell for $5 per lb and candy for $3 per lb. A young consumer has $10 to spend on peanuts and candy and has the utility function, $U(P, C) = 5 \ln(P) \ln(C)$.

How much candy and peanuts should our consumer purchase to maximize their utility subject to the $10 budget constraint?

Budget Constraint: $5P + 3C = 10 \implies P = 2 - 0.6C$

Internalized Utility Function: $\bar{U}(C) = 5 \ln(C) \ln(2 - 0.6C)$

First Order Conditions: $\bar{U}'(C) = -\frac{3 \ln(C)}{2 - 0.6C} + \frac{5 \ln(2 - 0.6C)}{C} = 0$

$\implies \frac{(10 - 3C) \ln(2 - 0.6C) - 3C \ln(C)}{2C - 0.6C^2} = 0$
Utility Max Problem Continued

For our problem on the previous slide, $C^* = 1.325 \text{ lbs}$ at cost $3.975.

Second Order Condition:

$U_{cc} = -\frac{3 \cdot (2 - 0.6C)/C + 1.8\ln(C)}{(2 - 0.6C)^2} + \frac{-3C/(2 - 0.6C) - 5\ln(2 - 0.6C)}{C^2}$

$\therefore$ maximum.

$U(C^*) = 5 \cdot \ln(1.325) \cdot \ln(1.205) = 5 \cdot (0.2814) \cdot (0.1865)$

$= 0.2624$
Graph of Utility Maximization
Utility Maximization

At point of utility maximization, the slope of the indifference curve equals the slope of the budget line, \(- \frac{P_C}{P_p} = -.6\)

From the implicit function theorem, we know the slope of our indifference curve is \(- \frac{U_C}{U_p} = - \frac{\ln(P)}{\ln(C)} \cdot \frac{P}{C}\)

For our example, \(C = 1.325, P = 1.205\)

Slope of indifference curve \(= - \frac{\ln(P)}{\ln(C)} \cdot \frac{P}{C} = - \frac{\ln(1.205) \cdot 1.205}{\ln(1.325) \cdot 1.325}\)

\(= - \frac{(0.1865) \cdot 1.205}{(0.2814) \cdot 1.325} = - \frac{0.2247}{0.37287} = 0.6026 \approx 0.6\)