# Economics 2301

Lecture 36

Constrained Optimization

# Constrained Optimization

- Much of economics analysis models choice in the face of trade-offs in one form or another.
- Constrained optimization problems appear in virtually every area of economics.
- The basic constrained optimization problem reflects a tension between what is desired and what is obtainable.
- The framework of any constrained optimization problem includes an objective function and one or more constraints.

# Constrained Optimization

- The **feasible set** of a constrained optimization problem is the set of arguments of the objective function that satisfies all the constraints of the problem simultaneously.
- **Equality constraints** are constraints that hold exactly.
- **Inequality constraints** allow a function of one or more of the choice variables to be less than or greater than some level.

# Solving Constrained Optimization Problems Through Substitution

- Constraints limit the feasible set of choices in an optimization problem.
- One way to solve a constrained optimization problem is to convert it to an unconstrained optimization problem by “internalizing” the constraint(s) directly into the objective function.
- The constraint is internalized when we express it as a function of one of the arguments of the objective function and then substitute out that argument using the constraint.

# Text Example

Find the maximum of the bivariate function

$$f(x_1, x_2) = 2\sqrt{x_1} + \frac{1}{2}\sqrt{x_2}$$

subject to the constraint : $4x_1 + x_2 = 20$

Now : $x_2 = 20 - 4x_1$

The internalized objective function is

$$\overline{f}(x_1) = 2\sqrt{x_1} + \frac{1}{2}\sqrt{20 - 4x_1}$$

First Order Condition : $\overline{f}'(x_1) = \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{20 - 4x_1}} = 0$

# Text Example Continued

$$\text{First Order Condition}: \overline{f}'(x_1) = \frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{20-4x_1}} = 0$$

$$\text{This equals zero } x_1 = 20 - 4x_1 \ or \ 20 - 5x_1 = 0$$

$$\Rightarrow x_1^* = 4 \ and \ x_2^* = 4$$

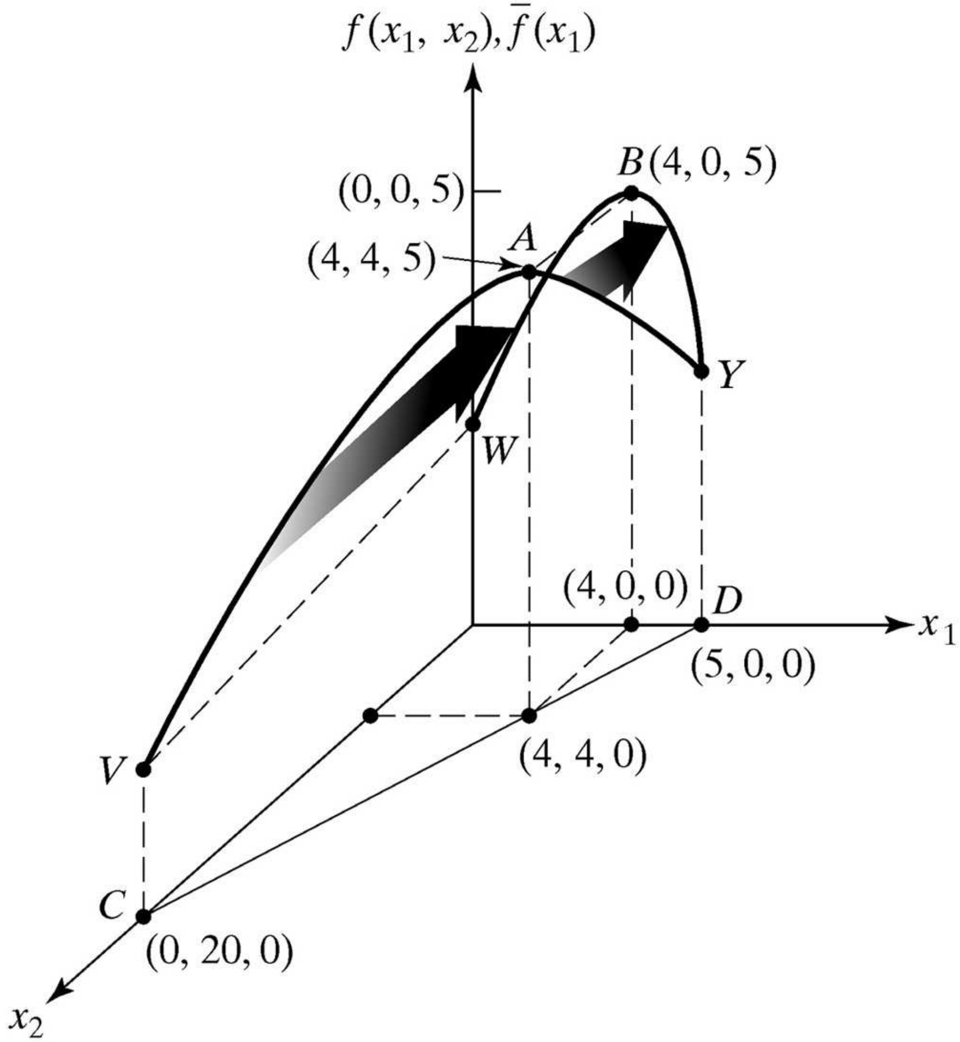
Second Order Condition :

$$\overline{f}''(x_1) = -\frac{1}{2} x_1^{-3/2} - 2 \cdot (20 - 4x_1)^{-3/2}$$

$$\text{at } x_1 = 4; \quad \overline{f}''(x_1) = -\frac{1}{2} \cdot \frac{1}{8} - 2 \cdot \frac{1}{8} = -\frac{5}{16} < 0$$

$\therefore$ The internalized function is maximized.

# Figure 11.1 Constrained Optimization and an Internalized Function



# Example 2

$$Let\ z = 20x - 2x^2 + 30y - 3y^2 + 4xy$$

$$Unconstrained\ Optimum:$$

$$1st\ Order\ conditions:$$

$$z_x = 20 - 4x + 4y = 0$$

$$z_y = 30 + 4x - 6y = 0$$

$$x^* = 30,\ y^* = 25,\ z^* = 675$$

$$\text{2nd Order Conditions}:\ z_{xx} = -4 < 0,\ z_{yy} = -6 < 0,\ z_{xy} = 4$$

$$z_{xx}z_{yy} = (-4)(-6) = 24 > 16 = 4^2 = z_{xy}^2$$

$$\therefore \text{a maximum.}$$

# Example 2 Continued

Now consider maximizing our function subject to the

constraint : $x + y = 40 \quad \Rightarrow y = 40\text{-}x$

Our internalized objective function is

$$\bar{z} = 20x - 2x^2 + 30(40 - x) - 3(40 - x)^2 + 4x(40 - x)$$

$$= 20x - 2x^2 + 1200 - 30x - 3(1600 - 80x + x^2) + 160x - 4x^2$$

$$= -3600 + 390x - 9x^2$$

First Order Condition : $\bar{z}_x = 390 - 18x = 0 \Rightarrow x^* = 21\frac{2}{3}, y^* = 18\frac{1}{3}$

Second Order Condition : $\bar{z}_{xx} = -18 < 0, \therefore$ maximum.

$\bar{z} = 625$

# Utility maximization Problem

Suppose peanuts sell for \$5 per lb and candy for \$3 per lb.

A young consumer has \$10 to spend on peanuts and candy and has the utility function, $U(P,C) = 5\ln(P)\ln(C)$.

How much candy and peanuts should our consumer purchase to maximize their utility subject to the \$10 budget constraint?

Budget Constraint : $5P + 3C = 10 \ \Rightarrow P = 2\text{-}0.6C$

Internalized Utility Function : $\overline{U}(C) = 5\ln(C)\ln(2 - 0.6C)$

First Order Conditions : $\overline{\mathrm{U}}'(C) = -\dfrac{3\ln(C)}{2-0.6C} + \dfrac{5\ln(2-0.6C)}{C} = 0$

$$\Rightarrow \frac{(10-3C)\ln(2-0.6C) - 3C\ln(C)}{2C - 0.6C^2} = 0$$

# Utility Max Problem Continued

For our problem on the previous slide, $C^* = 1.325\, lbs$

$at\ \cos t$ \$3.975.

Second Order Condition :

$$\overline{U}_{CC} = -\frac{3\cdot\left(2-0.6C\right)/C + 1.8\ln(C)}{\left(2-0.6C\right)^2} +$$

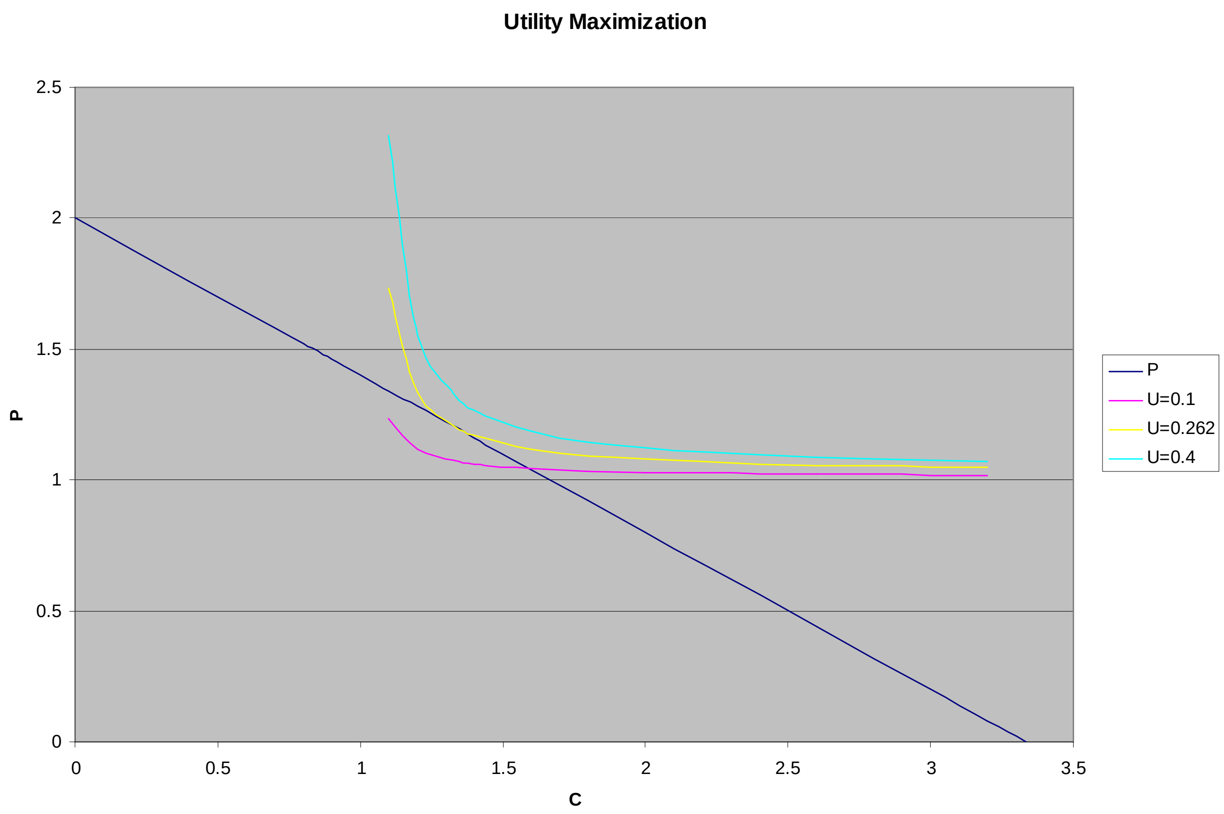
$$\frac{-3C/\left(2-0.6C\right) - 5\ln\left(2-0.6C\right)}{C^2} < 0$$

$\therefore$ maximum.

$$\overline{U}\left(C^*\right) = 5 * \ln(1.325)\ln(1.205) = 5\cdot\left(0.2814\right)\cdot\left(0.1865\right)$$

$$= 0.2624$$

# Graph of Utility Maximization



# Utility Maximization

At point of utility maximization, the slope of the indifference curve equals the slope of the budget line, $-P_C / P_P = -.6$

From the implicit function theorem, we know the slope of our indifference curve is $-U_C / U_P = -\frac{\ln(P)}{\ln(C)} \cdot \frac{P}{C}$

For our example, $C = 1.325$, $P = 1.205$

$$\text{Slope of indifference curve} = -\frac{\ln(P)}{\ln(C)} \cdot \frac{P}{C} = -\frac{\ln(1.205) \cdot 1.205}{\ln(1.325) \cdot 1.325}$$

$$= -\frac{(0.1865) \cdot 1.205}{(0.2814) \cdot 1.325} = -\frac{0.2247}{0.37287} = 0.6026 \approx 0.6$$