Economics 2301

Lecture 37

Constrained Optimization

Lagrange Multiplier Method

- An alternative to substituting a equality constraint into an objective function as a method for solving optimization problems is the **Lagrange multiplier method**.
- This involves forming a **Lagrangian function** that includes the objective function, the constraint, and a variable called a **Lagrange multiplier**.
- Problems are often easier to solve with the Lagrange multiplier method than with the substitution method.

Setting up and solving Lagrangian Functions

The Lagrangian function for a constrained maximization or a constrained minimization problem with the objective function $f(x_1,x_2)$ subject to the equality constraint that $g(x_1,x_2)=c$ is

$$L(x_1, x_2, \lambda) = f(x_1,x_2) - \lambda\left(g(x_1,x_2) - c\right)$$

where x_1 and x_2 are the two variables to be chosen. Arguments of the Lagrangian function include both the arguments of the objective function and the Lagrange Multiplier, λ. Also note that the value of the Lagrangian function is the same as the value of the objective function when the constraint holds.

Solving Lagrangian Functions

The first - order conditions for the Lagrangian function w ith respect to each of the arguments of the objective function and the equality constraint satisfy th e following equations with respect to the values of x_1, x_2, and λ:

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} = 0,$$

$$\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} = 0, \; and$$

$$g(x_1, x_2) = c.$$

Utility Maximization Example

The utility derived from exercise (X) and watching movies (M) is described by the function

$$U(X,M) = 100 - e^{-2X} - e^{-M}$$

Four hours per day are available to watch movies and exercise.

Our Lagrangian function is

$$L(X,M,\lambda) = 100 - e^{-2X} - e^{-M} - \lambda(X + M - 4)$$

First-Order Conditions:

$$L_X = 2e^{-2X} - \lambda = 0$$

$$L_M = e^{-M} - \lambda = 0$$

$$L_\lambda = X + M - 4 = 0$$

Utility Max example continued

First - Order Conditions :

$$L_X = 2e^{-2X} - \lambda = 0,\ L_M = e^{-M} - \lambda = 0,\ L_\lambda = X + M - 4 = 0$$

From (2) we get that $\lambda = e^{-M}$. Substituting into (1), we get

$2e^{-2X} - e^{-M} = 0$ Solving (3) for X and substituting, we get

$$2e^{-2X} - e^{-(4-X)} = 0 \rightarrow \ln(2) - 2X = -4 + X \text{ or } 3X = \ln(2) + 4$$

$$X^* = \frac{\ln(2)+4}{3} \ and\ M^* = \frac{8-\ln(2)}{3}$$

Interpreting Lagrange Multiplier

- With our Lagrangian function, we have a new variable, λ, the Lagrange multiplier.
- The Lagrange multiplier, λ, represents the effect of a small change in the constraint on the optimal value of the objective function.

Proof of Interpretation

Consider the optimal value of a constrained maximization problem in which the objective function is $f(x,y)$ and there is one constraint $g(x,y) = c$. Call the optimal value of the two arguments $x^*(c)$ and $y^*(c)$. At the optimum $g(x^*(c),y^*(c)) = c$. Using the chain rule we take the derivative of both sides of this expression.

$$\frac{\partial g(x^*(c),y^*(c))}{\partial x} \cdot \frac{dx^*(c)}{dc} + \frac{\partial g(x^*(c),y^*(c))}{\partial y} \cdot \frac{dy^*(c)}{dc} = 1$$

Use the chain rule to differentiate the optimal value of objective.

$$\frac{df\left(x^*(c),y^*(c)\right)}{dc} = \frac{\partial f(x^*(c),y^*(c))}{\partial x} \cdot \frac{dx^*(c)}{dc} + \frac{\partial f(x^*(c),y^*(c))}{\partial y} \cdot \frac{dy^*(c)}{dc}$$

Proof Continued.

The first - order conditions require

$$\frac{\partial f\left(x^*(c), y^*(c)\right)}{\partial x} = \lambda \frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial x} \; and$$

$$\frac{\partial f\left(x^*(c), y^*(c)\right)}{\partial y} = \lambda \frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial y} \quad substituting, we\ get$$

$$\frac{df\left(x^*(c), y^*(c)\right)}{dc} = \lambda \left[\frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial x} \frac{dx^*(c)}{dc} + \frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial y} \frac{dy^*(c)}{dc} \right]$$

The term in the bracket is equal to 1, hence

$$\frac{df\left(x^*(c), y^*(c)\right)}{dc} = \lambda$$

Utility Max Example

For example of watching movies and exercise,

$$\lambda^* = e^{-M} = e^{-\frac{8-\ln(2)}{3}} \approx e^{-2.4356} \approx 0.0875$$

Utility at the optimal values is

$$U^* = 100 - e^{-2X} - e^{-M} = 100 - e^{-3.1288} - e^{-2.4356}$$

$$\approx 100 - .0438 - .0875 = 99.8687$$

Suppose we have 5 hours for movies and exercise.

$$X^{**} = 1.8977, M^{**} = 3.1023, \lambda^{**} = 0.0449$$

$$U^{**} = 100 - e^{-2X} - e^{-M} \approx 100 - e^{-3.7954} - e^{-3.1023}$$

$$\approx 100 - 0.0225 - 0.0449 \approx 99.9326$$

$$\Delta U = U^{**} - U^* = 99.9326 - 99.8687 = 0.0639$$

$$\approx \frac{\lambda^* + \lambda^{**}}{2} = \frac{0.0875 + 0.0449}{2} = 0.0662$$