Economics 2301

Lecture 38

Constrained Optimization

Interpreting Lagrange Multiplier

- With our Lagrangian function, we have a new variable, λ, the Lagrange multiplier.
- The Lagrange multiplier, λ, represents the effect of a small change in the constraint on the optimal value of the objective function.

Proof of Interpretation

Consider the optimal value of a constrained maximization problem in which the objective function is $f(x,y)$ and there is one constraint $g(x,y) = c$. Call the optimal value of the two arguments $x^*(c)$ and $y^*(c)$. At the optimum $g(x^*(c), y^*(c)) = c$. Using the chain rule we take the derivative of both sides of this expression.

$$\frac{\partial g(x^*(c),y^*(c))}{\partial x}\cdot\frac{dx^*(c)}{dc}+\frac{\partial g(x^*(c),y^*(c))}{\partial y}\cdot\frac{dy^*(c)}{dc}=1$$

Use the chain rule to differentiate the optimal value of objective.

$$\frac{df\left(x^*(c),y^*(c)\right)}{dc}=\frac{\partial f(x^*(c),y^*(c))}{\partial x}\cdot\frac{dx^*(c)}{dc}+\frac{\partial f(x^*(c),y^*(c))}{\partial y}\cdot\frac{dy^*(c)}{dc}$$

Proof Continued.

The first - order conditions require

$$\frac{\partial f\left(x^*(c), y^*(c)\right)}{\partial x} = \lambda \frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial x} \; and$$

$$\frac{\partial f\left(x^*(c), y^*(c)\right)}{\partial y} = \lambda \frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial y} \quad substituting, we \; get$$

$$\frac{df\left(x^*(c), y^*(c)\right)}{dc} = \lambda \left[\frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial x} \frac{dx^*(c)}{dc} + \frac{\partial g\left(x^*(c), y^*(c)\right)}{\partial y} \frac{dy^*(c)}{dc} \right]$$

The term in the bracket is equal to 1, hence

$$\frac{df\left(x^*(c), y^*(c)\right)}{dc} = \lambda$$

Utility Max Example

For example of watching movies and exercise,

$$\lambda^* = e^{-M} = e^{-\frac{8-\ln(2)}{3}} \approx e^{-2.4356} \approx 0.0875$$

Utility at the optimal values is

$$U^* = 100 - e^{-2X} - e^{-M} = 100 - e^{-3.1288} - e^{-2.4356}$$

$$\approx 100 - .0438 - .0875 = 99.8687$$

Suppose we have 5 hours for movies and exercise.

$$X^{**} = 1.8977,\ M^{**} = 3.1023,\ \lambda^{**} = 0.0449$$

$$U^{**} = 100 - e^{-2X} - e^{-M} \approx 100 - e^{-3.7954} - e^{-3.1023}$$

$$\approx 100 - 0.0225 - 0.0449 \approx 99.9326$$

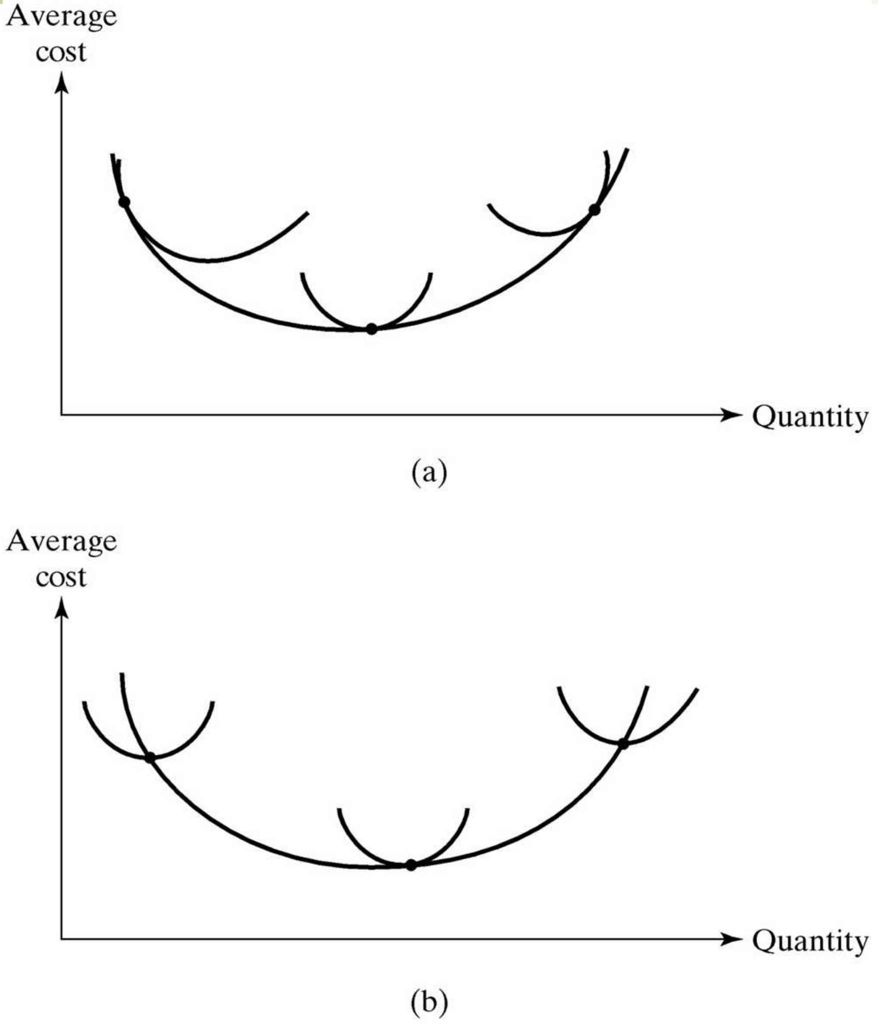
$$\Delta U = U^{**} - U^* = 99.9326 - 99.8687 = 0.0639$$

$$\approx \frac{\lambda^* + \lambda^{**}}{2} = \frac{0.0875 + 0.0449}{2} = 0.0662$$

Interpreting the Lagrange Multiplier

In other contexts, the Lagrange multiplier may be interpreted differently. For example, if the objective function represents the profit function from undertaking an activity and the constraint reflects a limit on using an input to that activity, the Lagrange multiplier reflects the marginal benefit from having additional input. In this case the Lagrange multiplier represents the price a firm would be willing to pay per unit of additional input, which is known as the **shadow price** of the input.

Figure 11.3 Short-Run Cost and Long-Run Cost Functions



Envelope Theorem

We have the objective function, $f(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m)$

subject to the constraint $g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m) = 0$

The x_i are choice variables and the β_j are parameters.

The Lagrangian is

$$L(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m) = f(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m)$$
$$- \lambda g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m)$$

The first-order conditions are

$$f_{x_i} - \lambda g_{x_i} = 0 \quad \text{for } i = 1,2,\ldots,n \ \text{ and}$$

$$g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m) = 0$$

Envelope Theorem Continued

Given a solution, we define the maximum value function as

$$F(\beta_1,\cdots,\beta_m) = f\left(x_1^*(\beta_1,\cdots,\beta_m),\cdots,x_n^*(\beta_1,\cdots,\beta_m);\beta_1,\cdots,\beta_m\right),$$

The chain rule shows that the derivative of the maximum value function with respect to the parameter β_j is

$$F_{\beta_j} = f_{x_1}\frac{dx_1^*}{d\beta_j}+\cdots+f_{x_n}\frac{dx_n^*}{d\beta_j}+f_{\beta_j}$$

The first-order condition $f_{x_i} = \lambda g_{x_i}$ for $i=1,\ldots,n$, can be used to show that, at the optimum, the derivative of the maximum value function is

$$F_{\beta_j} = \lambda g_{x_1}\frac{dx_1^*}{d\beta_j}+\cdots+\lambda g_{x_n}\frac{dx_n^*}{d\beta_j}+f_{\beta_j}$$

$$=\lambda\left(g_{x_1}\frac{dx_1^*}{d\beta_j}+\cdots+g_{x_n}\frac{dx_n^*}{d\beta_j}\right)+f_{\beta_j}$$

Envelope Theorem Continued

Differenti ating the constraint evaluated at the optimal level of the variables , in a manner analogous to previous slide, gives us

$$g_{x_1}\frac{dx_1^*}{d\beta_j}+\cdots+g_{x_n}\frac{dx_n^*}{d\beta_j}+g_{\beta_j}=0$$

or, equivalent ly, the term in the parenthese s in the second line of the derivative of the maximum value function is

$$\left(g_{x_1}\frac{dx_1^*}{d\beta_j}+\cdots+g_{x_n}\frac{dx_n^*}{d\beta_j}\right)=-g_{\beta_j}$$

Substituti ng this expression into the derivative of the maximum value function gives us the general statement of the Envelope Theorem $:F_{\beta_j}=f_{\beta_j}-\lambda g_{\beta_j}$.

Expressing this in terms of the Lagrangian function for the constraine d optimizati on problem $:F_{\beta_j}=L_{\beta_j}|_{x_1,\ldots,x_n}$.

Envelope Theorem Conclusion

- The envelope theorem shows that the effect of a small change in a parameter of a constrained optimization problem on its maximum value can be determined by considering only the partial derivative of the objective function and the partial derivative of the constraint with respect to that parameter.
- To a first approximation, it is not necessary to consider how a small change in a parameter affects the optimal value of the variables of the problem in order to evaluate the change in its maximum value.

Average Cost Curves

Our short-run cost minimization problem is

$$L(N,\overline{K},\lambda,r,w,Y)) = -\left[wN + r\overline{K} - \lambda\left(f\left(N,\overline{K}\right) - Y\right)\right]$$

The maximum value function for our short-run problem is

$C\left(\overline{K}, r, w, Y\right)$ and its partial derivative with respect to output is

$$C_Y = \frac{\partial L(N^*,\overline{K},\lambda,r,w,Y)}{\partial Y}\Big|_N = -\lambda$$

The Lagrangian for the long-run problem is

$$L(N,K,\lambda,r,w,Y)) = -\left[wN + rK - \lambda\left(f\left(N,K\right) - Y\right)\right]$$

The maximum value function is C(w, r, Y).

Average Cost Function cont.

The partial derivative of the long-run maximum value function

is $C_Y = \frac{\partial L\left(N^*, K^*, \lambda\lambda, r, w, \right)}{\partial Y}\Big|_{N,K} = -\lambda$

This is the same as the partial derivative of the short-run maximum value function with respect to Y. This implies the slope of the long-run total cost curve equals the slope of the short-run total cost curve with the fixed and variable level of capital. Hence, the same is true for the average cost curves.

Figure 11.3 Short-Run Cost and Long-Run Cost Functions