Economics 2301

Lecture 39

Constrained Optimization
Figure 11.3 Short-Run Cost and Long-Run Cost Functions
Envelope Theorem

We have the objective function, \( f(x_1, \ldots, x_n, \beta_1, \ldots, \beta_m) \)
subject to the constraint \( g(x_1, \ldots, x_n, \beta_1, \ldots, \beta_m) = 0 \)
The \( x_i \) are choice variables and the \( \beta_j \) are parameters.
The Lagrangian is
\[
L(x_1, \ldots, x_n, \beta_1, \ldots, \beta_m) = f(x_1, \ldots, x_n, \beta_1, \ldots, \beta_m) - \lambda g(x_1, \ldots, x_n, \beta_1, \ldots, \beta_m)
\]
The first-order conditions are
\[
f_{x_i} - \lambda g_{x_i} = 0 \quad \text{for } i = 1, 2, \ldots, n \quad \text{and}
g(x_1, \ldots, x_n, \beta_1, \ldots, \beta_m) = 0
\]
Given a solution, we define the maximum value function as
\[ F(\beta_1, \ldots, \beta_m) = f(x_1^*(\beta_1, \ldots, \beta_m), \ldots, x_n^*(\beta_1, \ldots, \beta_m); \beta_1, \ldots, \beta_m), \]
The chain rule shows that the derivative of the maximum value function with respect to the parameter \( \beta_j \) is
\[
F_{\beta_j} = f_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + f_{x_n} \frac{dx_n^*}{d\beta_j} + f_{\beta_j}
\]
The first-order condition \( f_{x_i} = \lambda g_{x_i} \) for \( i = 1, \ldots, n \), can be used to show that, at the optimum, the derivative of the maximum value function is
\[
F_{\beta_j} = \lambda g_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + \lambda g_{x_n} \frac{dx_n^*}{d\beta_j} + f_{\beta_j}
\]
\[
= \lambda \left( g_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + g_{x_n} \frac{dx_n^*}{d\beta_j} \right) + f_{\beta_j}
\]
Differentiating the constraint evaluated at the optimal level of the variables, in a manner analogous to previous slide, gives us

\[ g_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + g_{x_n} \frac{dx_n^*}{d\beta_j} + g_{\beta_j} = 0 \]

or, equivalently, the term in the parentheses in the second line of the derivative of the maximum value function is

\[ \left( g_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + g_{x_n} \frac{dx_n^*}{d\beta_j} \right) = -g_{\beta_j} \]

Substituting this expression into the derivative of the maximum value function gives us the general statement of the Envelope Theorem: \( F_{\beta_j} = f_{\beta_j} - \lambda g_{\beta_j} \).

Expressing this in terms of the Lagrangian function for the constrained optimization problem: \( F_{\beta_j} = L_{\beta_j} \mid_{x_1, \ldots, x_n} \).
The envelope theorem shows that the effect of a small change in a parameter of a constrained optimization problem on its maximum value can be determined by considering only the partial derivative of the objective function and the partial derivative of the constraint with respect to that parameter.

To a first approximation, it is not necessary to consider how a small change in a parameter affects the optimal value of the variables of the problem in order to evaluate the change in its maximum value.
Average Cost Curves

Our short-run cost minimization problem is
\[ L(N, K, \lambda, r, w, Y)) = -[wN + rK - \lambda f(N, K) - Y] \]
The maximum value function for our short-run problem is
\( C(K, r, w, Y) \) and its partial derivative with respect to output is
\[ C_Y = \frac{\partial L(N^*, K, \lambda, r, w, Y)}{\partial Y} \bigg|_{N} = -\lambda \]
The Lagrangian for the long-run problem is
\[ L(N, K, \lambda, r, w, Y)) = -[wN + rK - \lambda f(N, K) - Y] \]
The maximum value function is \( C(w, r, Y) \).
The partial derivative of the long-run maximum value function is \( C_Y = \frac{\partial L(N^*, K^*, \lambda, r, w, \ldots)}{\partial Y} \bigg|_{N,K} = -\lambda \)

This is the same as the partial derivative of the short-run maximum value function with respect to \( Y \). This implies the slope of the long-run total cost curve equals the slope of the short-run total cost curve with the fixed and variable level of capital. Hence, the same is true for the average cost curves.
Figure 11.3 Short-Run Cost and Long-Run Cost Functions

(a) Short-Run Cost Function

(b) Long-Run Cost Function
We have a competitive firm that sells an output at price, $p$. It uses to inputs $x_1$ and $x_2$ that they purchase competitively at prices $w_1$ and $w_2$. Their production function is $f(x_1, x_2)$. The Lagrangian function for profit maximization is

$$L(x_1, x_2, y, p, w_1, w_2, \lambda) = py - w_1 x_1 - w_2 x_2 - \lambda (f(x_1, x_2) - y)$$

The Envelope Theorem shows that the derivatives of the maximum value profit function, $\Pi(p, w_1, w_2)$, evaluated at the optimal values of the choice variables $y^*, x_1^*$, and $x_2^*$ are
Envelope and Profit Maximizing continued

\[
\frac{\partial \Pi(p, w_1, w_2)}{\partial p} = \frac{\partial L(x^*_1, x^*_2, y^*, p, w_1, w_2, \lambda)}{\partial p} \bigg|_{x_1, x_2 = y^* > 0}
\]

\[
\frac{\partial \Pi(p, w_1, w_2)}{\partial w_1} = \frac{\partial L(x^*_1, x^*_2, y^*, p, w_1, w_2, \lambda)}{\partial w_1} \bigg|_{x_1, x_2 = -x^*_1 < 0}
\]

\[
\frac{\partial \Pi(p, w_1, w_2)}{\partial w_2} = \frac{\partial L(x^*_1, x^*_2, y^*, p, w_1, w_2, \lambda)}{\partial w_2} \bigg|_{x_1, x_2 = -x^*_2 < 0}
\]

These results are known as Hotelling's Lemma.
Constrained Optimization Example

Becker's optimal allocation of time model made utility a function of consuming goods, not the goods themselves, \( U(A_1, A_2, \ldots, A_n) \). The \( i \)th activity requires \( G_i = A_i \cdot n_i \) units of a good and \( t_i = A_i \cdot h_i \) hours of time, where \( n_i \) and \( h_i \) are units of goods and units of time, respectively, per one unit of activity, \( i \). Productivity advances imply less good to complete an activity, \( n_i \) decreases. Time-saving advances lower the time-cost of the \( i \)th activity, which is reflected by a decrease in \( h_i \). Consider a world of two activities. The total time, \( T \), is spent in work, \( W \), in activity 1, \( t_1 \), or in activity 2, \( t_2 \). Our constraint is \( T = W + t_1 + t_2 \).
Financial Constraint. Income equal wage times hours worked, $wW$. This is spent on goods 1 and 2 at prices $P_1$ and $P_2$. The Constraint is $wW = P_1G_1 + P_2G_2$. Defining the money value of total time as $wT$, we can write a combined constraint as $wT = P_1G_1 + P_2G_2 + wt_1 + wt_2$. We can rewrite the constraint in terms of activities as $wT = P_1n_1A_1 + P_2n_2A_2 + wh_1A_1 + wh_2A_2$. The utility function, $U(A_1, A_2)$ is assumed to be monotonically increasing in each of its arguments and to have strictly negative second partial derivatives.
The Lagrangian function for our example is

\[ L(A_1, A_2, \lambda) = U(A_1, A_2) - \lambda [(P_1n_1 + wh_1)A_1 + (P_2n_2 + wh_2)A_2 - wT] \]

First-order conditions:

\[ L_{A_1} = U_1(A_1, A_2) - \lambda (P_1n_1 + wh_1) = 0 \]

\[ L_{A_2} = U_2(A_1, A_2) - \lambda (P_2n_2 + wh_2) = 0 \]

\[ L_\lambda = -[(P_1n_1 + wh_1)A_1 + (P_2n_2 + wh_2)A_2 - wT] = 0 \]

From the first two equations, we get

\[ \frac{U_1(A_1, A_2)}{U_2(A_1, A_2)} = \frac{P_1n_1 + wh_1}{P_2n_2 + wh_2} \]

ratio marginal utilities = ratio costs.
The effect of a change in the wage rate on the ratio is
\[
\frac{\partial}{\partial w} \left( \frac{(P_1 n_1 + w h_1)}{(P_2 n_2 + w h_2)} \right)
\]
\[
= \frac{(P_2 n_2 + w h_2) h_1 - (P_1 n_1 + w h_1) h_2}{(P_2 n_2 + w h_2)^2} > 0 \quad \text{when}
\]
\[
h_1 > \left[ \frac{(P_1 n_1 + w h_1)}{(P_2 n_2 + w h_2)} \right] h_2
\]