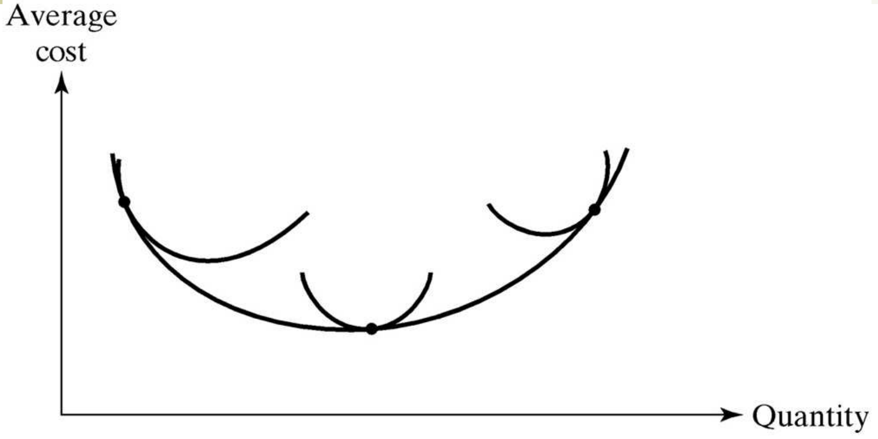
Economics 2301

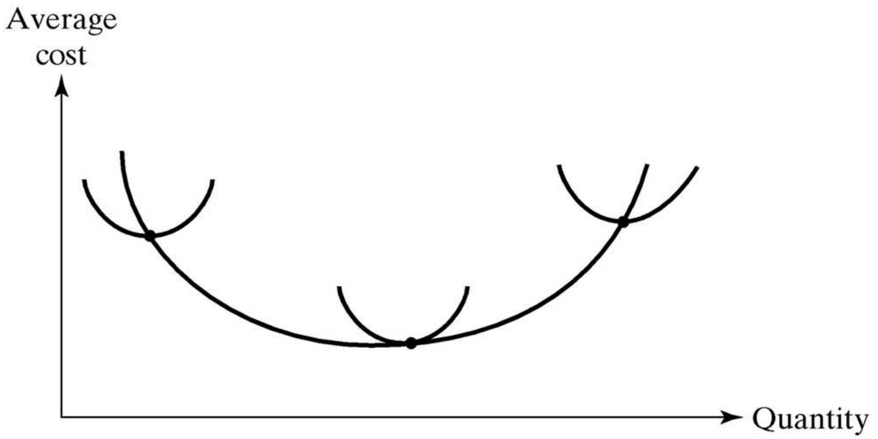
Lecture 39

Constrained Optimization

Figure 11.3 Short-Run Cost and Long-Run Cost Functions



(a)



(b)

Envelope Theorem

We have the objective function, $f(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m)$

subject to the constraint $g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m) = 0$

The x_i are choice variables and the β_{j} are parameters.

The Lagrangian is

$$L(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m) = f(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m)$$
$$- \lambda g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m)$$

The first-order conditions are

$$f_{x_i} - \lambda g_{x_i} = 0 \quad for\ i = 1,2,\ldots,n \ \ and$$

$$g(x_1, \cdots, x_n, \beta_1, \cdots, \beta_m) = 0$$

Envelope Theorem Continued

Given a solution, we define the maximum value function as

$$F(\beta_1, \cdots, \beta_m) = f\left(x_1^*(\beta_1, \cdots, \beta_m), \cdots, x_n^*(\beta_1, \cdots, \beta_m); \beta_1, \cdots, \beta_m\right),$$

The chain rule shows that the derivative of the maximum value function with respect to the parameter β_j is

$$F_{\beta_j} = f_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + f_{x_n} \frac{dx_n^*}{d\beta_j} + f_{\beta_j}$$

The first-order condition $f_{x_i} = \lambda g_{x_i}$ for $i = 1, \ldots, n$, can be used to show that, at the optimum, the derivative of the maximum value function is

$$F_{\beta_j} = \lambda g_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + \lambda g_{x_n} \frac{dx_n^*}{d\beta_j} + f_{\beta_j}$$

$$= \lambda\left(g_{x_1} \frac{dx_1^*}{d\beta_j} + \cdots + g_{x_n} \frac{dx_n^*}{d\beta_j} \right) + f_{\beta_j}$$

Envelope Theorem Continued

Differentiating the constraint evaluated at the optimal level of the variables, in a manner analogous to previous slide, gives us

$$g_{x_1}\frac{dx_1^*}{d\beta_j}+\cdots+g_{x_n}\frac{dx_n^*}{d\beta_j}+g_{\beta_j}=0$$

or, equivalently, the term in the parentheses in the second line of the derivative of the maximum value function is

$$\left(g_{x_1}\frac{dx_1^*}{d\beta_j}+\cdots+g_{x_n}\frac{dx_n^*}{d\beta_j}\right)=-g_{\beta_j}$$

Substituting this expression into the derivative of the maximum value function gives us the general statement of the Envelope Theorem $: F_{\beta_j} = f_{\beta_j} - \lambda g_{\beta_j}$.

Expressing this in terms of the Lagrangian function for the constrained optimization problem $: F_{\beta_j} = L_{\beta_j}|_{x_1,\ldots,x_n}$.

Envelope Theorem Conclusion

- The envelope theorem shows that the effect of a small change in a parameter of a constrained optimization problem on its maximum value can be determined by considering only the partial derivative of the objective function and the partial derivative of the constraint with respect to that parameter.
- To a first approximation, it is not necessary to consider how a small change in a parameter affects the optimal value of the variables of the problem in order to evaluate the change in its maximum value.

Average Cost Curves

Our short-run cost minimization problem is

$$L(N,\overline{K},\lambda,r,w,Y)) = -\left[wN + r\overline{K} - \lambda\left(f\left(N,\overline{K}\right) - Y\right)\right]$$

The maximum value function for our short-run problem is

$C\left(\overline{K},r,w,Y\right)$ and its partial derivative with respect to output is

$$C_Y = \frac{\partial L(N^*,\overline{K},\lambda,r,w,Y)}{\partial Y}\Big|_N = -\lambda$$

The Lagrangian for the long-run problem is

$$L(N,K,\lambda,r,w,Y)) = -\left[wN + rK - \lambda\left(f\left(N,K\right) - Y\right)\right]$$

The maximum value function is C(w, r, Y).

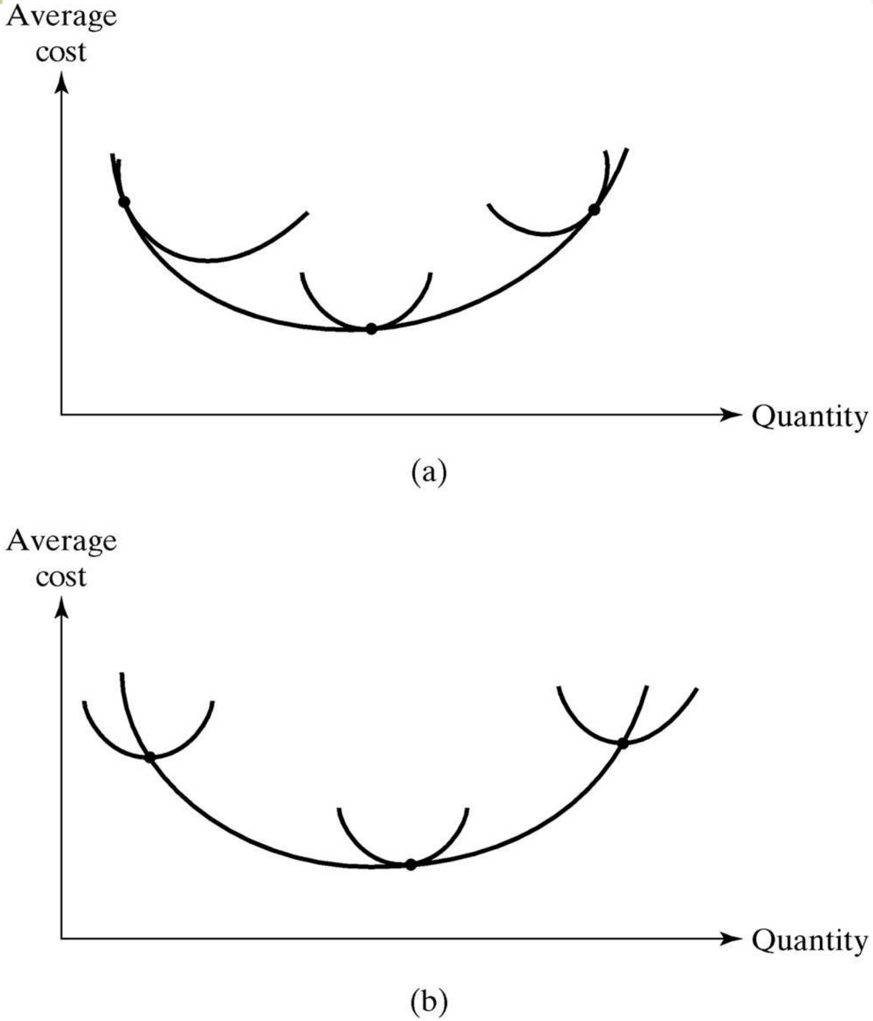
Average Cost Function cont.

The partial derivative of the long-run maximum value function

is $C_Y = \dfrac{\partial L\left(N^*, K^*, \lambda\lambda, r, w, \right)}{\partial Y}\Big|_{N,K} = -\lambda$

This is the same as the partial derivative of the short-run maximum value function with respect to Y. This implies the slope of the long-run total cost curve equals the slope of the short-run total cost curve with the fixed and variable level of capital. Hence, the same is true for the average cost curves.

Figure 11.3 Short-Run Cost and Long-Run Cost Functions



Envelope and Profit Maximizing

We have a competive firm that sells an output at price, *p*. It uses to inputs x_1 and x_2 that they purchase competively at prices w_1 and w_2. Their production function is $f(x_1,x_2)$. The Lagrangian function for profit maximization is

$$L(x_1,x_2,y,p,w_1,w_2,\lambda)= py - w_1x_1 - w_2x_2 - \lambda\left(f(x_1,x_2) - y\right)$$

The Envelope Theorem shows that the derivatives of the maximum value profit funciton, $\Pi(p,w_1,w_2)$, evaluated at the optimal values of the choice variables $y^*, x_1^*, and\ x_2^*\ are$

Envelope and Profit Maximizing continued

$$\frac{\partial \Pi\left(p, w_1, w_2\right)}{\partial p} = \frac{\partial L\left(x_1^*, x_2^*, y^*, p, w_1, w_2, \lambda\right)}{\partial p}\Big|_{x_1, x_2} = y^* > 0$$

$$\frac{\partial \Pi\left(p, w_1, w_2\right)}{\partial w_1} = \frac{\partial L\left(x_1^*, x_2^*, y^*, p, w_1, w_2, \lambda\right)}{\partial w_1}\Big|_{x_1, x_2} = -x_1^* < 0$$

$$\frac{\partial \Pi\left(p, w_1, w_2\right)}{\partial w_2} = \frac{\partial L\left(x_1^*, x_2^*, y^*, p, w_1, w_2, \lambda\right)}{\partial w_2}\Big|_{x_1, x_2} = -x_2^* < 0$$

These results are known as Hotelling's Lemma.

Constrained Optimization Example

Becker's optimal allocation of time model made utility a function of consuming goods, not the goods themselves, $U(A_1, A_2, \ldots, A_n)$. The ith activity requires $G_i = A_i \cdot n_i$ units of a good and $t_i = A_i \cdot h_i$ hours of time, where n_i and h_i are units of goods and units of time, respectively, per one unit of activity, i. Productivity advances imply less good to complete an activity, n_i decreases. Time-saving advances lower the time-cost of the *ith* activity, which is reflected by a decrease in h_i. Consider a world of two activities. The total time, T, is spent in work, W, inctivity 1, t_1, or in activity 2, t_2. Our constraint is $T = W + t_1 + t_2$.

Becker Example Continued

Financial Constraint. Income equal wage times hours worked, wW. This is spent on goods 1 and 2 at prices P_1 and P_2. The Constraint is $wW = P_1G_1 + P_2G_2$. Defining the money value of total time as wT, we can write a combined constraint as $wT = P_1G_1 + P_2G_2 + wt_1 + wt_2$. We can rewrite the constraint in terms of activities as

$$wT = P_1n_1A_1 + P_2n_2A_2 + wh_1A_1 + wh_2A_2.$$

The utility function, $U(A_1, A_2)$ is assumed to be monotonically increasing in each of its arguments and to have strictly negative second partial derivatives.

Becker Example Continued

The Lagrangian function for our example is

$$L(A_1, A_2, \lambda) = U(A_1, A_2) - \lambda[(P_1 n_1 + wh_1)A_1 + (P_2 n_2 + wh_2)A_2 - wT]$$

First-order conditions:

$$L_{A_1} = U_1(A_1, A_2) - \lambda(P_1 n_1 + wh_1) = 0$$

$$L_{A_2} = U_2(A_1, A_2) - \lambda(P_2 n_2 + wh_2) = 0$$

$$L_{\lambda} = -[(P_1 n_1 + wh_1)A_1 + (P_2 n_2 + wh_2)A_2 - wT] = 0$$

From the first two equations, we get

$$\frac{U_1(A_1, A_2)}{U_2(A_1, A_2)} = \frac{P_1 n_1 + wh_1}{P_2 n_2 + wh_2}, \text{ ratio marginal utilities} = \text{ratio costs.}$$

Becker Example Continued

The effect of a change in the wage rate on the ratio is

$$\frac{\partial\left(\left(P_1 n_1 + wh_1\right)/\left(P_2 n_2 + wh_2\right)\right)}{\partial w}$$

$$= \frac{\left(P_2 n_2 + wh_2\right)h_1 - \left(P_1 n_1 + wh_1\right)h_2}{\left(P_2 n_2 + wh_2\right)^2} > 0 \quad when$$

$$h_1 > \left[\left(P_1 n_1 + wh_1\right)/\left(P_2 n_2 + wh_2\right)\right]h_2$$