Economics 2301

Lecture 40
Constrained Optimization
More Advanced Work

3 Choice Variables 1 Constraint

$$Optimize \quad f(x, y, z) = \ln(x+y+z) \; with \; the \; constraint \quad xyz = 64$$

$$Lagrangian \quad Function$$

$$L(x, y, z, \lambda) = \ln(x+y+z) - \lambda(xyz - 64)$$

$$First \quad Order \quad Conditions$$

$$\frac{\partial L}{\partial x} = \frac{1}{x+y+z} - \lambda yz = 0$$

$$\frac{\partial L}{\partial y} = \frac{1}{x+y+z} - \lambda xz = 0$$

$$\frac{\partial L}{\partial z} = \frac{1}{x+y+z} - \lambda xy = 0$$

$$\frac{\partial L}{\partial \lambda} = xyz - 64 = 0$$

Solving Problem

$$\textit{Multipling first } 3 \textit{ equations together gives}$$

$$\frac{1}{(x+y+z)^3}=\lambda^3(xyz)^2$$

$$\frac{1}{(x+y+z)^3}=\lambda^3(64)^2$$

$$\lambda^3=\frac{1}{64^2(x+y+z)^3}$$

$$\lambda=\frac{1}{16(x+y+z)}$$

$$\textit{substitute equation } 1$$

$$\frac{1}{(x+y+z)}-\frac{yz}{16(x+y+z)}=0$$

$$\frac{16-yz}{16(x+y+z)}=0$$

Solving Problem cont.

$$By \quad symmetry \quad we \quad get$$
$$16 - yz = 0$$
$$16 - xz = 0$$
$$16 - xy = 0$$
$$line\,1 \rightarrow y = \frac{16}{z}$$
$$substitute \quad this \quad result \quad into \quad equation \quad 3 \quad gives$$
$$16 - \frac{16\text{x}}{z} = 0$$
$$\rightarrow 16\text{z} - 16\text{x} = 0$$
$$By \quad symmetry \quad we \quad get \quad x = y = z$$
$$\rightarrow \quad equation \quad 1\, gives \quad x^2 = 16$$
$$x = y = z = 4$$

2^{nd} Order Conditions

- **Sufficient Condition for a Maximum with one constraint:** Consider a Lagrangian function consisting of an objective function with *n* arguments and one constraint. If the determinant of the bordered Hessian of that Lagrangian function evaluated at a stationary point has the same sign as $(-1)^n$ and the largest n-1 leading principal minors alternate in sign, then the quadratic form is negative definite on the constraint, and the stationary point represents a maximum.

2nd Order Condition

- **Sufficient Condition for a Minimum with one constraint:** Consider a Lagrangian function consisting of an objective function with *n* arguments and one constraint. If all of determinants of the *n*-1 leading principal minors of its bordered Hessian evaluated at the stationary point are negative, including the determinant of bordered Hessian itself, then the quadratic form is positive definite on the constraint, an the stationary point represents a minimum.

Bordered Hessian our Problem

$$\bar{H} = \begin{bmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & L_{11} & L_{12} & L_{13} \\ g_2 & L_{21} & L_{22} & L_{23} \\ g_3 & L_{31} & L_{32} & L_{33} \end{bmatrix}$$

Bordered Hessian

$$\bar{H} = \begin{bmatrix} 0 & yz & xz & xy \\ yz & \dfrac{-1}{(x+y+z)^2} & \dfrac{-1}{(x+y+z)^2} - \lambda z & \dfrac{-1}{(x+y+z)^2} - \lambda y \\ xz & \dfrac{-1}{(x+y+z)^2} - \lambda z & \dfrac{-1}{(x+y+z)^2} & \dfrac{-1}{(x+y+z)^2} - \lambda x \\ xy & \dfrac{-1}{(x+y+z)^2} - \lambda y & \dfrac{-1}{(x+y+z)^2} - \lambda x & \dfrac{-1}{(x+y+z)^2} \end{bmatrix}$$

2nd Order Conditions

$$\bar{H}=\begin{bmatrix} 0 & yz & xz & xy \\ yz & \frac{-1}{(x+y+z)^2} & \frac{-1}{(x+y+z)^2}-\frac{z}{16(x+y+z)} & \frac{-1}{(x+y+z)^2}-\frac{y}{16(x+y+z)} \\ xz & \frac{-1}{(x+y+z)^2}-\frac{z}{16(x+y+z)} & \frac{-1}{(x+y+z)^2} & \frac{-1}{(x+y+z)^2}-\frac{x}{16(x+y+z)} \\ xy & \frac{-1}{(x+y+z)^2}-\frac{y}{16(x+y+z)} & \frac{-1}{(x+y+z)^2}-\frac{x}{16(x+y+z)} & \frac{-1}{(x+y+z)^2} \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 16 & 16 & 16 \\ 16 & \frac{-1}{144} & \frac{-1}{36} & \frac{-1}{36} \\ 16 & \frac{-1}{36} & \frac{-1}{144} & \frac{-1}{36} \\ 16 & \frac{-1}{36} & \frac{-1}{36} & \frac{-1}{144} \end{bmatrix}$$

2nd Order Conditions

$$Number\ \ principal\ \ minors\ \ needed\ \ be\ \ evaluate\ =(n-1)=(3-1)=2$$

$$|\bar{H}|=\begin{vmatrix} 0 & 16 & 16 & 16 \\ 16 & \frac{-1}{144} & \frac{-1}{36} & \frac{-1}{36} \\ 16 & \frac{-1}{36} & \frac{-1}{144} & \frac{-1}{36} \\ 16 & \frac{-1}{36} & \frac{-1}{36} & \frac{-1}{144} \end{vmatrix} = -0.333$$

$$|\bar{H}_1|=\begin{vmatrix} 0 & 16 & 16 \\ 16 & \frac{-1}{144} & \frac{-1}{36} \\ 16 & \frac{-1}{36} & \frac{-1}{144} \end{vmatrix} = -10.67$$

$$Therefore,\ \ we\ \ have\ \ a\ \ ???$$

General Optimization Problem

$$\textit{Optimize} \quad f(x_1, x_2, \ldots, x_n) \quad \textit{subject} \quad p \quad (p<n) \quad \textit{constraints}$$

$$g_1(x_1, x_2, \ldots, x_n) = c_1 \quad \ldots \quad g_p(x_1, x_2, \ldots, x_n) = c_p$$

$$\textit{Lagraniam} \quad \textit{Function}$$

$$L(x_1, x_2, \ldots, x_n, \lambda_1, \ldots, \lambda_p) = f(x_1, x_2, \ldots, x_n) - \lambda_1(g_1(x_1, x_2, \ldots, x_n) - c_1) - \ldots. - \lambda_p(g_p(x_1, x_2, \ldots, x_n) - c_p)$$

$$\text{1st} \quad \textit{Order} \quad \textit{Conditions}$$

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial g_1}{\partial x_1} - \cdots - \lambda_p \frac{\partial g_p}{x_1} = 0$$

$$\vdots$$

$$\frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \lambda_1 \frac{\partial g_1}{\partial x_n} - \cdots - \lambda_p \frac{\partial g_p}{x_n} = 0$$

$$\frac{\partial L}{\partial \lambda_1} = g_1(x_1, x_2, \ldots, x_n) - c_1 = 0$$

$$\vdots$$

$$\frac{\partial L}{\partial \lambda_p} = g_P(x_1, x_2, \ldots, x_n) - c_p = 0$$

Bordered Hessian

$$\bar{H} = \begin{bmatrix} 0 & \cdots & 0 & \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_1} & L_{11} & \cdots & L_{1\mathrm{n}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_p}{\partial x_n} & L_{n1} & \cdots & L_{nn} \end{bmatrix}$$

Sufficient Condition for Constrained Maximum

- Consider a Lagrangian function consisting of an objective function with n arguments and p $(p<n)$ constraints. If the determinant of the bordered Hessian of that Lagrangian function evaluated at a stationary point has the same sign as $(-1)^n$ and the largest $n-p$ leading principal minors alternate in sign, then the quadratic form is negative definite on the set of constraints, and the stationary point represents a maximum.

Sufficient Condition for a Constrained Minimum

- Consider a Lagrangian function consisting of an objective function with *n* arguments and *p* *($p<n$)* constraints. If all of the largest *n*-*p* leading principal minors of its bordered Hessian evaluated at the stationary point have the sign $(-1)^p$, including the determinant of the bordered Hessian itself, then the quadratic form is positive definite on the set of constraints, and the stationary point represents a minimum.