Economics 2301

Lecture 41
Constrained Optimization
Advanced Work 2

Maximization Problem

$$Maximize\ f(x, y, z) = xyz \ \ subject\ g_1(x, y, z) = x^2 + y^2 = 1 \ ; \ g_2(x, y, z) = x + z = 1$$

$$Lagrangian \ \ Function$$

$$L(x, y, z, \lambda_{1,} \lambda_2) = f(x, y, z) - \lambda_1(g_1(x, y, z) - c_1) - \lambda_2(g_2(x, y, z) - c_2)$$

$$L(x, y, z, \lambda_{1,} \lambda_2) = xyz - \lambda_1(x^2 + y^2 - 1) - \lambda_2(x + z - 1)$$

$$First \ \ Order \ \ Conditions$$

$$\frac{\partial L}{\partial x} = yz - \lambda_1(2\mathrm{x}) - \lambda_2 = 0$$

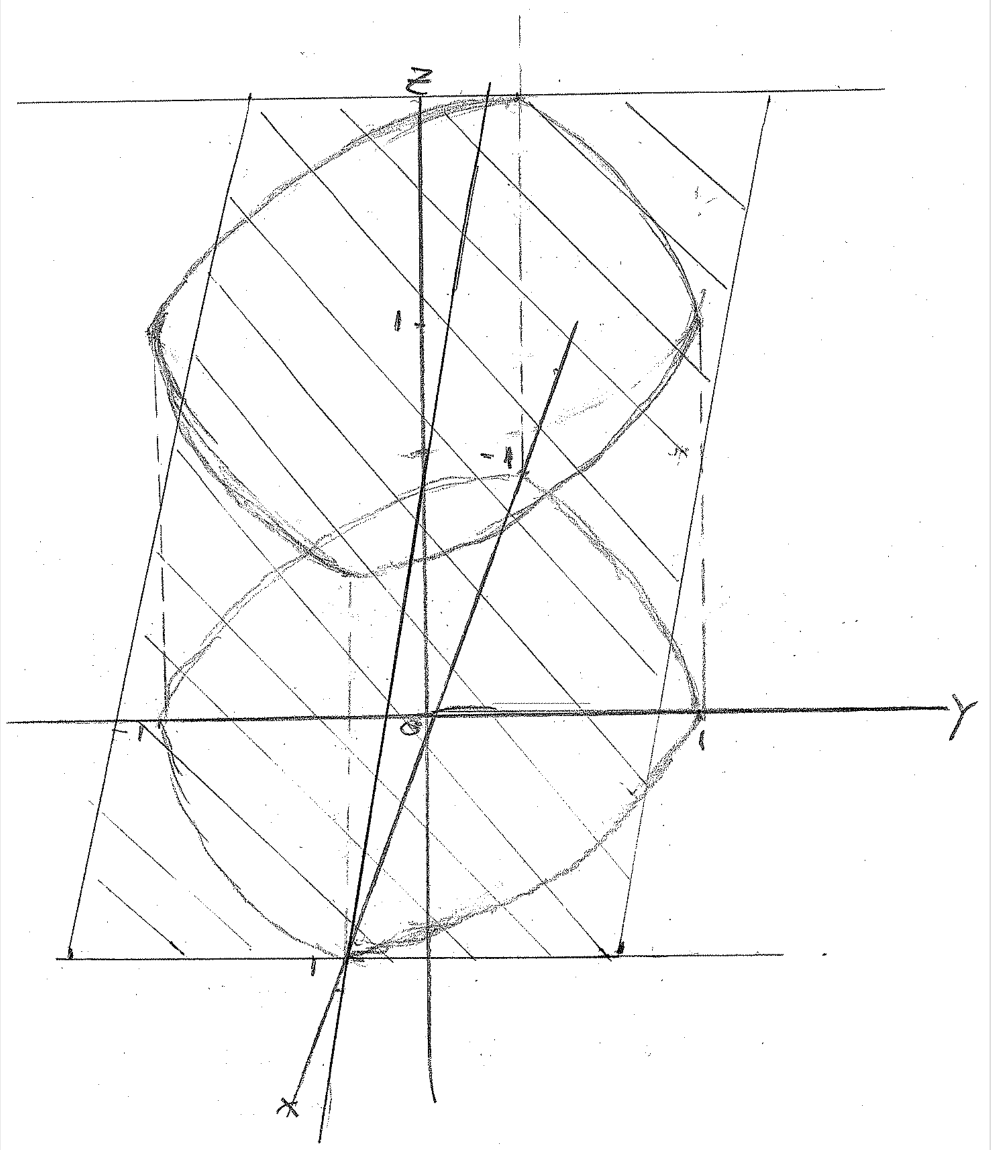
$$\frac{\partial L}{\partial y} = xz - \lambda_1(2\mathrm{y}) = 0$$

$$\frac{\partial L}{\partial z} = xy - \lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = x^2 + y^2 - 1 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = x + z - 1 = 0$$

Drawing of Problem



Problem Continued

$$Solve\ \ second\ \ \text{and}\ \ third\ \ equations\ \ for\ \ \lambda_1\ \ \text{and}\ \ \lambda_2$$

$$\lambda_1 \ = \ \frac{xz}{2\text{y}}$$

$$\lambda_2 \ = \ xy$$

$$Substitutes\ \ these\ \ into\ \ equation\ \ 1$$

$$yz-2\left(\frac{xz}{2\text{y}}\right)-xy \ = \ 0$$

$$\rightarrow y^2z-x^2z-xy^2 \ = \ 0$$

$$solve\ \ \text{4th}\ \ equation\ \ for\ \ y^2\ \ \text{and}\ \ \text{5th}\ \ equation\ \ for\ \ z$$

$$y^2=1-x^2\ \ \text{and}\ \ z=1-x$$

$$insert\ \ these\ \ into\ \ equation\ \ above,\ \ which\ \ gives$$

$$(1-x^2)(1-x)-x^2(1-x)-x(1-x^2) \ = \ 0$$

$$= \ (1-x)[(1-x^2)-x^2-x(1+x)] \ = \ 0$$

$$one\ \ root\ \ is\ \ 1\ \ the\,others\ \ are\ \ the\ \ solution\ \ of\ \ the\ \ quadratic\ \ equation$$

$$3\text{x}^2+x-1 \ = \ 0$$

$$The\ \ roots\ \ are\,x=0.4343\ \ \text{and}\ \ -0.7676$$

$$using\ \ equations\ \ 4\ \ \text{and}\ \ 5\ \ gives$$

$$x\approx 0.4343 \quad y\approx \pm 0.9008 \quad z\approx 0.5657$$

$$x\approx -0.7676 \quad y\approx \pm 0.6409 \quad z\approx 1.7676$$

$$The\ \ max\ \ is\ \ at\ \ x\approx -0.7676 \quad y\approx -0.6409 \quad z\approx 1.7676$$

Sufficient Condition for Constrained Maximum

- Consider a Lagrangian function consisting of an objective function with n arguments and p $(p<n)$ constraints. If the determinant of the bordered Hessian of that Lagrangian function evaluated at a stationary point has the same sign as $(-1)^n$ and the largest n-p leading principal minors alternate in sign, then the quadratic form is negative definite on the set of constraints, and the stationary point represents a maximum.

Sufficient Condition for a Constrained Minimum

- Consider a Lagrangian function consisting of an objective function with *n* arguments and *p* ($p<n$) constraints. If all of the largest *n-p* leading principal minors of its bordered Hessian evaluated at the stationary point have the sign $(-1)^p$, including the determinant of the bordered Hessian itself, then the quadratic form is positive definite on the set of constraints, and the stationary point represents a minimum.

Bordered Hessian

$$\bar{H} = \begin{bmatrix} 0 & 0 & \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ 0 & 0 & \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_1}{\partial x} & \frac{\partial g_2}{\partial x} & L_{xx} & L_{xy} & L_{xz} \\ \frac{\partial g_1}{\partial y} & \frac{\partial g_2}{\partial y} & L_{yx} & L_{yy} & L_{yz} \\ \frac{\partial g_1}{\partial z} & \frac{\partial g_2}{\partial z} & L_{zx} & L_{zy} & L_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2\text{x} & 2\text{y} & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 2\text{x} & 1 & 2\lambda_1 & z & y \\ 2\text{y} & 0 & z & 2\lambda_1 & x \\ 0 & 1 & y & x & 0 \end{bmatrix}$$

Bordered Hessian Cont.

$$\bar{H} = \begin{bmatrix} 0 & 0 & 2\text{x} & 2\text{y} & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 2\text{x} & 1 & -2\lambda_1 & z & y \\ 2\text{y} & 0 & z & -2\lambda_1 & x \\ 0 & 1 & y & x & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1.5352 & -1.2818 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ -1.5352 & 1 & -2.1170 & 1.7676 & -0.6409 \\ -1.2818 & 0 & 1.7676 & -2.1170 & -0.7676 \\ 0 & 1 & -0.6409 & -0.7676 & 0 \end{bmatrix}$$

$$For \quad our \quad problem \quad n=3 \quad \text{and} \quad p=2.$$

$$For \quad max \quad we \quad need \quad \text{to} \quad evaluate \quad n-p=3-2=1 \quad principal \quad minors$$

$$sign \quad must \; = \; (-1)^n \; = \; (-1)^3 \; = \; -1$$

$$|\bar{H}| \; = \; -16.2295$$

$$Therefore, \quad we \quad have \quad a \quad maximum.$$

Utility Maximization: Logarithmic Utility

$$Maximize \quad U = \ln(x) + \ln(y) \quad subject \quad P_x x + P_y y = I$$

$$Lagrangian \quad Function: \quad L(x, y, \lambda) = \ln(x) + \ln(y) - \lambda(P_x x + P_y y - I)$$

$$First \quad Order \quad Conditions:$$

$$\frac{\partial L}{\partial x} = \frac{1}{x} - \lambda P_x = 0$$

$$\frac{\partial L}{\partial y} = \frac{1}{y} - \lambda P_y = 0$$

$$\frac{\partial L}{\partial \lambda} = P_x x + P_y y - I = 0$$

$$Eliminating \quad \lambda \quad \text{from} \quad the \quad first \quad two \quad first-order \quad conditions \quad yields$$

$$-\frac{1}{x} / \frac{1}{y} = -\frac{P_x}{P_y}$$

$$marginal \quad rate \quad of \quad substitution \quad between \quad x \text{ and } y \quad equals \quad their \quad price \quad ratios.$$

2^{nd} Order Conditions

$$\bar{H} = \begin{bmatrix} 0 & P_x & P_y \\ P_x & -\frac{1}{x^2} & 0 \\ P_y & 0 & -\frac{1}{y^2} \end{bmatrix}$$

$$|\bar{H}| = \frac{P_y^2}{x^2} + \frac{P_x^2}{y^2} > 0$$

$$\rightarrow maximum$$

Deriving Demand X

$$Solve \quad 2\text{nd} \quad first-order \quad condition \quad for \quad \lambda$$

$$\lambda = \frac{1}{(P_y y)}$$

$$\text{from} \quad 3\text{rd} \quad first-order-condition$$

$$P_y y \;=\; I - P_x x \quad substituting \quad we \quad get$$

$$\lambda = \frac{1}{(I - P_x x)}$$

$$substituting \quad into \quad 1\text{st} \quad first-order \quad condition \quad gives$$

$$\frac{1}{x} = \left(\frac{1}{(I - P_x x)} \right) P_x \quad \text{or}$$

$$x = \frac{I}{P_x} - x \ \text{or}$$

$$x = \frac{I}{2\text{P}_x}$$

Time, Money Constraint

Consider a consumer who consumes n goods. The consumption of these goods also requires time for each unit consumed. The consumer has a budget constraint and a time constraint.

$$\text{Maximize}\, U(x_{1,} x_{2,.} .., x_n) \;\; \text{subject to} \;\; p_1 x_1 + ... + p_n x_n = I \;\; \text{and} \;\; t_1 x_1 + ... + t_n x_n = T$$

$$\text{Lagrangian Function is}$$

$$L(x_{1,} ..., x_n, \lambda_1, \lambda_2) = U(x_{1,} x_{2,.} .., x_n) - \lambda_1 (p_1 x_1 + ... + p_n x_n - I) - \lambda_2 (t_1 x_1 + ... + t_n x_n - T)$$

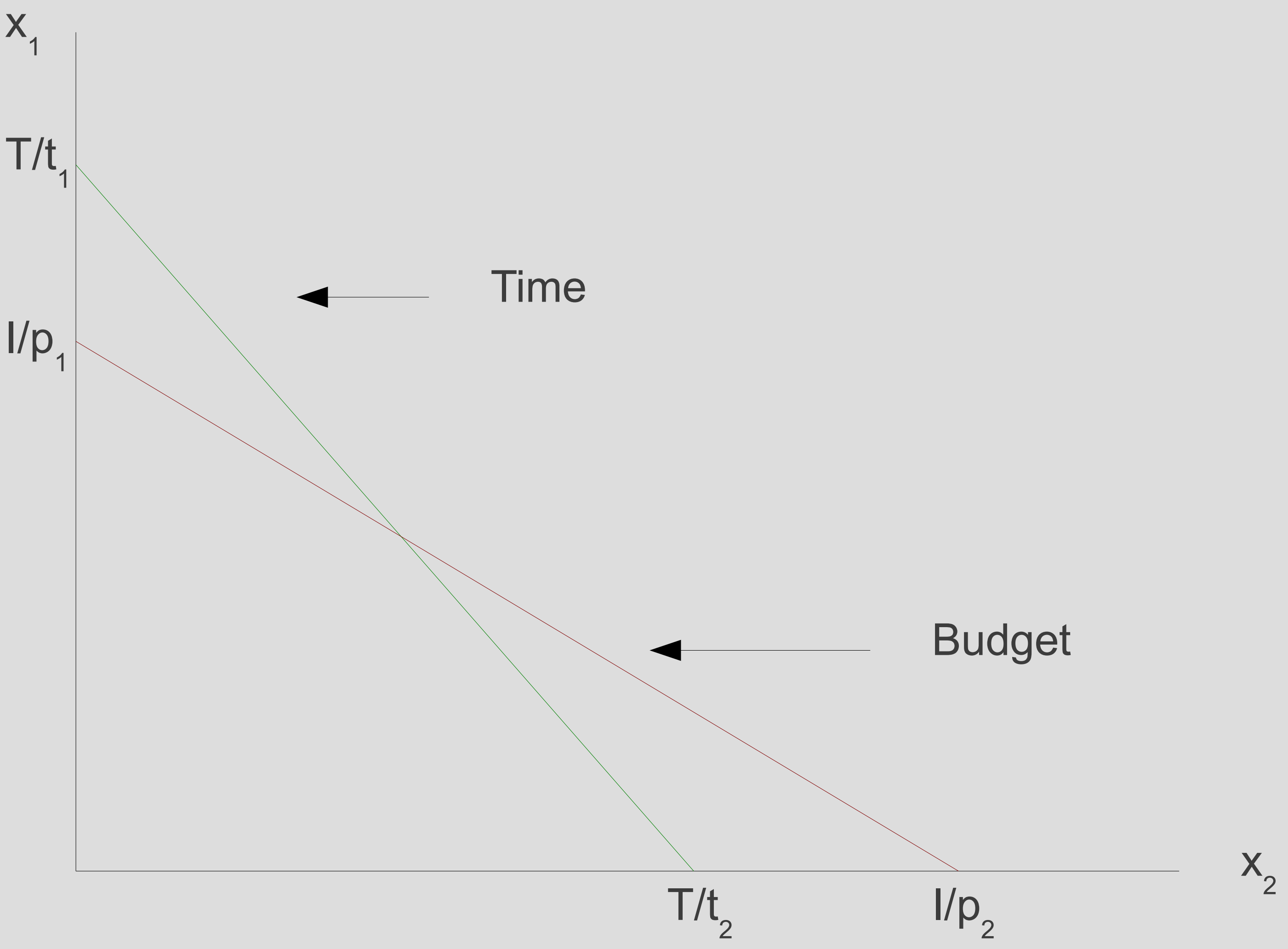
$$\text{First-Order Conditions}$$

$$\frac{\partial L}{\partial x_i} = \frac{\partial U}{\partial x_i} - \lambda_1 p_i - \lambda_2 t_i = 0 \qquad i = 1, ..., n$$

$$\frac{\partial L}{\partial \lambda_1} = p_1 x_1 + ... + p_n x_n - I = 0$$

$$\frac{\partial L}{\partial \lambda_2} = t_1 x_1 + ... + t_n x_n - T = 0$$

Constraints Picture



Problem Continued

$$\text{Bordered Hessian for the problem}$$

$$\bar{H} = \begin{bmatrix} 0 & 0 & p_1 & \cdots & p_n \\ 0 & 0 & t_1 & \cdots & t_n \\ p_1 & t_1 & L_{11} & \cdots & L_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_n & t_n & L_{n1} & \cdots & L_{nn} \end{bmatrix}$$

$$\text{This bordered Hessian satisfies the conditions for a maximum.}$$

Solving Problem

$$Using\ 2\ of\ the\ first\ n\ first-order\ conditions,\ we\ get$$

$$\frac{U_i}{U_j} = \frac{(\lambda_1 p_i + \lambda_2 t_i)}{(\lambda_1 p_j + \lambda_2 t_j)}$$

Left *hand side is marginal rate of substitution.*

Define $\tilde{\lambda} = \frac{\lambda_2}{\lambda_1}$, λ_1 *is marginal utility of income.* λ_2 *is marginal utility of time.*

$$\tilde{\lambda} = \left(\frac{utility}{time}\right) / \left(\frac{utility}{dollars}\right) = \frac{dollars}{time}$$

Now we can rewrite the marginal rate of substitution as

$$\frac{U_i}{U_j} = \frac{(p_i + \tilde{\lambda} t_i)}{(p_j + \tilde{\lambda} t_j)}$$