Economics 2301

Lecture 6

Growth Rates

Regional Growth in the U.S.

	Year			Annual Growth Rate	
Region	**1998**	**2003**	**2008**	1998-2003	2003-2008
New England	497,756	612,006	763,683	4.2%	4.5%
Mideast	1,613,291	2,013,636	2,605,113	4.5%	5.3%
Great Lakes	1,421,599	1,683,064	1,983,039	3.4%	3.3%
Plains	578,616	713,213	910,517	4.3%	5.0%
Southeast	1,901,498	2,409,888	3,148,037	4.9%	5.5%
Southwest	892,049	1,171,729	1,698,748	5.6%	7.7%
Rocky Mountain	267,870	348,184	482,328	5.4%	6.7%
Far West	1,506,978	1,934,451	2,574,100	5.1%	5.9%

New England Income

In general, we have the relation

$X_{t+1} = (1+r)X_t \quad where\ X_t = income\ in\ year\ t$

$r = annual\ growth\ rate\ in\ decimal\ form.$

New England Income

1999

$X_{1999} = (1+r)X_{1998} = (1+.042)X_{1998}$

$= 1.042(497756) = 518662$

2000

$X_{2000} = (1+r)X_{1999} = (1+r)\left[(1+r)X_{1998}\right]$

$= (1+r)(1+r)X_{1998} = (1+r)^2 X_{1998}$

$= (1.042)^2(497756) = 1.085764(497756) = 540446$

Change

$X_{1999} - X1998 = 20{,}906 \quad X_{2000} - X_{1999} = 21784$

Growth Formula

Growth Formula with Discrete End-of-Period Compounding

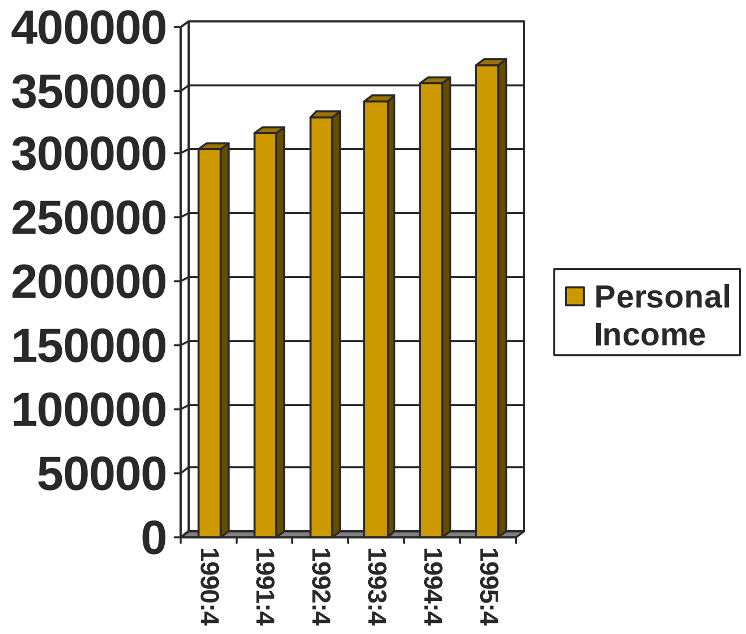
The relationship between the value of a variable in period, t, X_t, and its level in period $t+n$ when it grows by the rate r at the end of each period, is

$$X_{t+n} = (1+r)^n X_t \quad \textit{where}$$

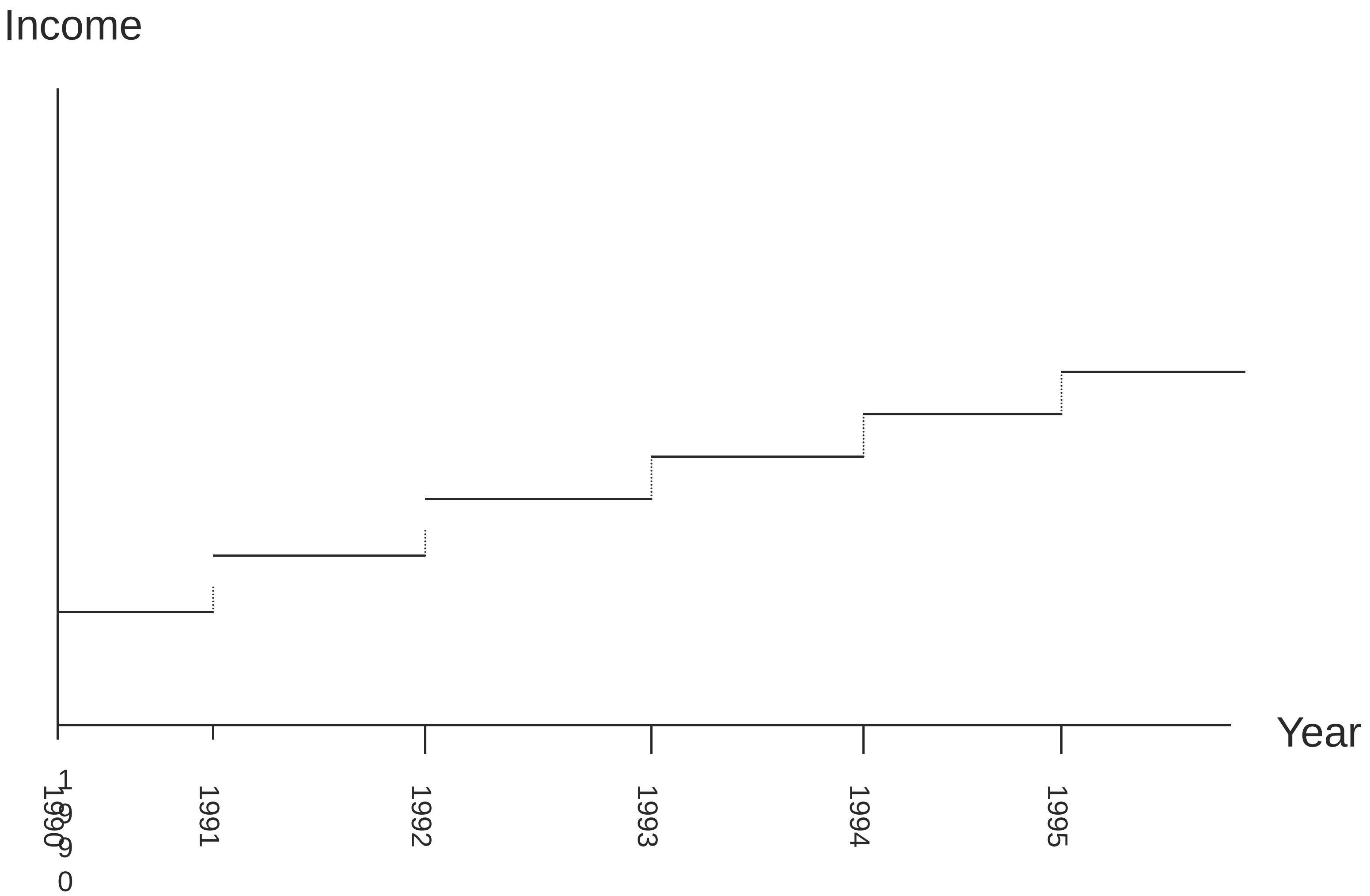
r is expressed in decimal form.

New England Personal Income

- Annual Growth Rate = 4%
- Our bar Chart is approximately a step function.
- We assume growth doesn't occur until end of the year.



Step Function



Income Growth

- We have depicted the growth of income over time as a step function.
- It is usually more natural to think of **continuous growth,** which would be reflected in a smooth evolution over time of variables like income and population.

Multiple Compounding in one Period

The relationship between the value of a variable at the beginning of period t, X_t, and its value at the beginning of the next period, X_{t+1}, when it grows by the rate r compounded k times during the period, is

$$X_{t+1} = \left(1 + \frac{r}{k}\right)^k X_t$$

Effect of Number of compounds at 4%

Compounds	$ 1,000.00
1	$ 1,040.00
2	$ 1,040.40
4	$ 1,040.60
12	$ 1,040.74
365	$ 1,040.81
Continuous	$ 1,040.81

Effect of Number of Compounds at 4%

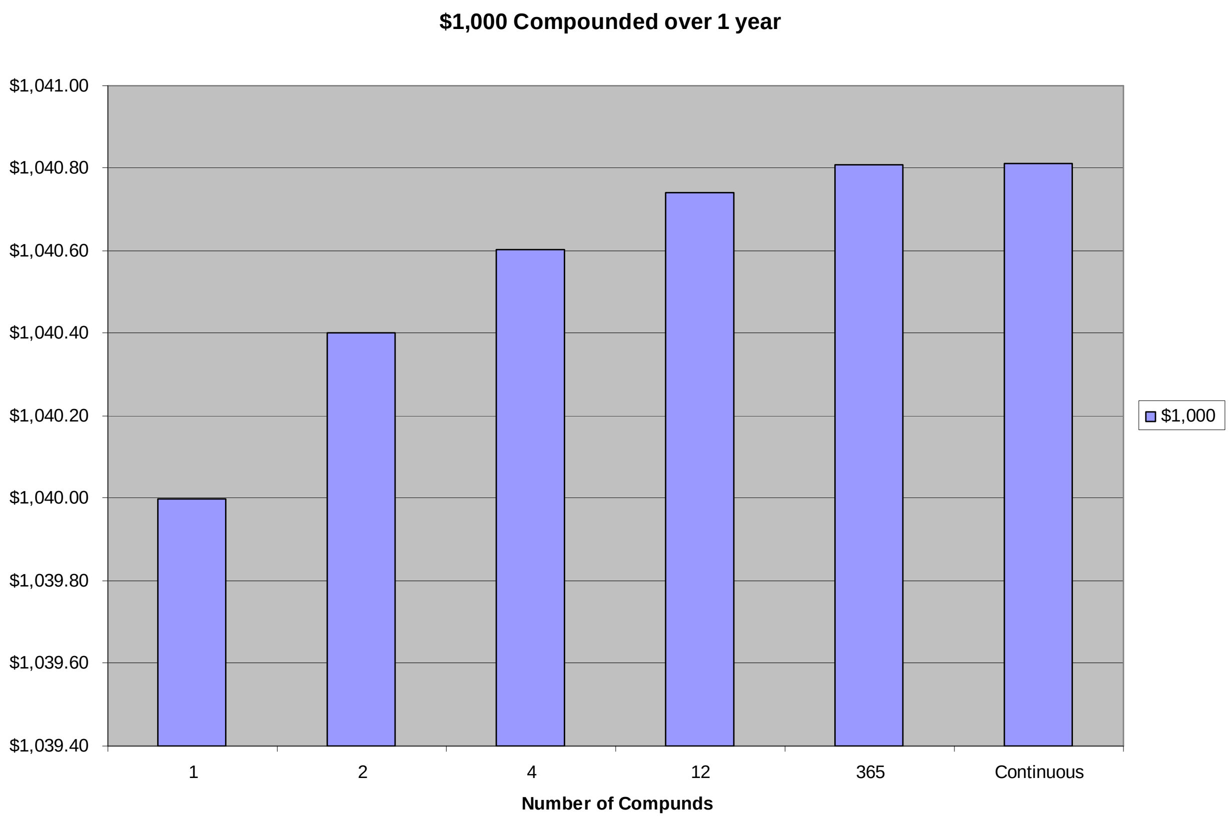
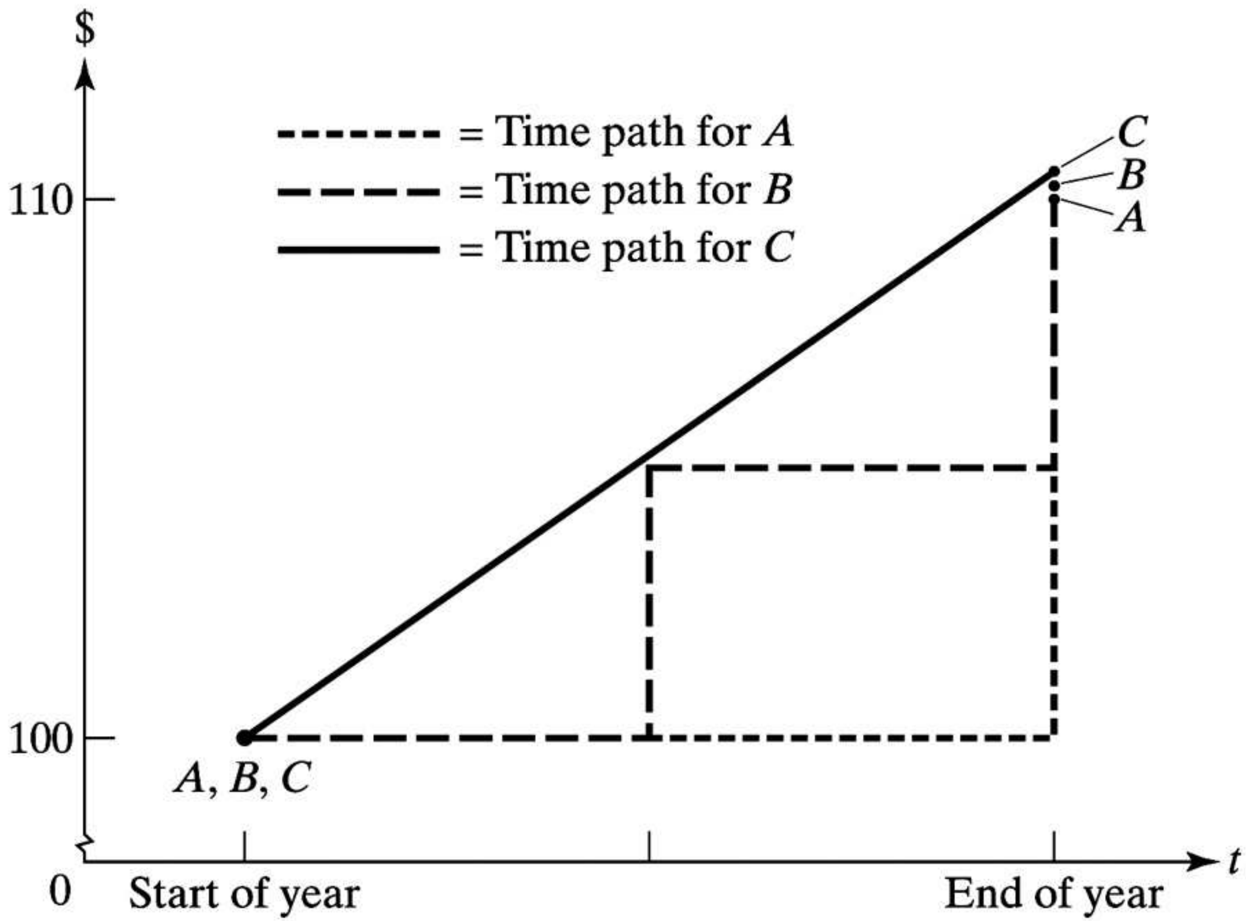


Figure 3.2 Compounding at Different Frequencies



Continuous Compounding The Exponential

From our example, we see that $(1+r/k)^k$ increases at a decreasing rate with an increase in k.

$$\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = e = 2.71828182845...$$

This is an irrational number. We assumed $r = 100\%$.

Continuous Compounding less 100% growth rate

$$\left(1+\frac{r}{k}\right)^{k}=\left(\left(1+\frac{r}{k}\right)^{k/r}\right)^{r}=\left(\left(1+\frac{1}{k/r}\right)^{k/r}\right)^{r}$$

note that as $k \to \infty$, $k/r \to \infty$

Let $m = k/r$

$$\lim_{k/r\to\infty}\left(\left(1+\frac{1}{k/r}\right)^{k/r}\right)^{r}=\lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^{m}\right)^{r}=e^{r}$$

1 Year continuous compounding

With continuous compounding and an annual growth rate of r, the amount we have at the end of the year, X_{t+1}, when we start the year with the amount, X_t, is

$$X_{t+1} = e^r X_t$$

Comparison of continuous and discreet compounding

Rate	Discreet	Continuous
2%	$ 1,020.00	$1,020.20
4%	$ 1,040.00	$1,040.81
6%	$ 1,060.00	$1,061.84
8%	$ 1,080.00	$1,083.29
10%	$ 1,100.00	$1,105.17

Continuous Compounding

The value of a variable X at the moment $t+n$ (that is $X(t+n)$), given its value at the moment t (that is, $X(t)$), when there is a continuous compound growth (or interest) rate of r per period, is

$$X(t+n) = X(t)e^{r \cdot n}$$

Effect of Continuous Compounding

$1000 with continuous Compounding for 10 years				
	Rate		**Amount**	
	2%		$ 1,221.40	
	4%		$ 1,491.82	
	6%		$ 1,822.12	
	8%		$ 2,225.54	
	10%		$ 2,718.28	

Annual and Effective Interest Rate

The annual interest rate represents the interest rate used in continuous compounding calculations. The effective interest rate is the interest rate that would give the same end-of-period return if interest were compounded only once at the end of the period. Let r_A be the annual rate and r_E be the effective rate, then we have $\$X(1+r_E) = \$Xe^{r_A}$ *or*

$$r_E = e^{r_A} - 1$$

Effective Interest Rates

Annual Rate		Effective Rate
2%		2.020%
4%		4.081%
6%		6.184%
8%		8.329%
10%		10.517%