**Economics 2301** **Name____________________________________**
**Quiz 7** **Fall 2010**

**Definitions or Equations (10 pts)**

**Lagrangian Function**

**The Lagrangian function for an optimization problem with *n* choice variables and *p* constraints can be written as**

$$L(x_1, \dots, x_n, \lambda_1, \dots, \lambda_p) = f(x_1, \dots, x_n) - \lambda_1(g_1(x_1, \dots, x_n) - c_1) - \cdots - \lambda_p(g_p(x_1, \dots, x_n) - c_p)$$

**Where $f(x_1, \dots, x_n)$ is the objective function to be optimized, and $g_i(x_1, \dots, x_n) = c_i,\ i = 1, \dots, p$ are p constraints on the optimization.**

**Sufficient Condition for a maximum in the Bivariate Case with one constraint**

**The determinant of the border Hessian $\bar{H} = \begin{bmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{yx} & L_{yy} \end{bmatrix}$ is positive when evaluated at the stationary point.**

**Generalized Sufficient Condition for a constrained minimum (n variables, p constraints)**

**Consider a Lagrangian function consisting of an objective function with n arguments and m constraints. If all of the largest n-m leading principal minors of its bordered Hessian evaluated at the stationary point have the sign $(-1)^m$, including the determinant of the bordered Hessian itself, then the quadratic form is positive definite on the set of constraints, and the stationary point represents a minimum.**

**Interpretation of the Lagrange Multiplier**

**The Lagrange multiplier represents the marginal impact on the objective function of a change in the constraint it represents, i.e. $\lambda = \frac{df(x^*(c), y^*(c))}{dc}$.**

**Simple statement of the basic conclusion of the envelope theorem**

**For a simple optimization problem with objective function $f(x_1, \dots, x_n; \beta_1, \dots, \beta_m)$ subject to a constraint $g(x_1, \dots, x_n; \beta_1, \dots, \beta_m)$ and the Lagrangian function $L(x_1, \dots, x_n; \beta_1, \dots, \beta_m) = f(x_1, \dots, x_n; \beta_1, \dots, \beta_m) - \lambda(g(x_1, \dots, x_n; \beta_1, \dots, \beta_m) - c)$ , and a optimization function $F(\beta_1, \dots, \beta_m)$, we have $F_{\beta_i} = f_{\beta_i} - \lambda g_{\beta_i}$, where $F_{\beta_i} = L_{\beta_i}|_{x_1, \dots, x_n}$.**

**Problem (10 Pts)**

**Suppose it is late in the semester and you have two exams left. You must decide how to allocate your working time during the study period. After eating, sleeping, exercising and maintaining some human contact, you have 12 hours each day in which to study for your exams. You have figured out that your grade point average GPA, from your two courses, Mathematical Methods and Literary Methods, takes the form**

$$GPA = \frac{2}{3}\left(\sqrt{M} + \sqrt{2L}\right),$$

**Where M is the number of hours per day spent studying for Mathematical Methods and L is the number of hours per day spent studying for Literary Methods. What is the optimal number of hours per day spent studying for each course? If you follow this strategy, what will your GPA be? Prove you have maximized your GPA.**

**Solution 1 using internalization**

**From the constraint we get M=12-L, substituting this into the objective function, we get** $\overline{GPA} = \frac{2}{3}(\sqrt{12-L} + \sqrt{2L})$**, the first order condition is then** $\frac{d\overline{GPA}}{dL} = \frac{2}{3}\left(-\frac{1}{2}(12-L)^{-\frac{1}{2}} + (2L)^{-\frac{1}{2}}\right) = 0.$

**From this we get** $\frac{1}{\sqrt{2L}} - \frac{1}{2\sqrt{12-L}} = 0 \; or \; 2\sqrt{12-L} - \sqrt{2L} = 0 \; or \; 4(12-L) = 2L \; or \; 6L = 48 \; which$

$$implies\; L^* = 8 \; and \; M^* = 4. \; At\; the\; optimum\; GPA = \frac{2}{3}\left(\sqrt{M} + \sqrt{2L}\right) = \frac{2}{3}\left(\sqrt{4} + \sqrt{16}\right)$$

$$= \frac{2}{3}(2+4) = \frac{2}{3}(6) = 4.$$

**Our second derivative of the internalized objective function is**

$\frac{d^2\overline{GPA}}{dL^2} = \frac{2}{3}\left[-\frac{1}{4}(12-L)^{-3/2} - (2L)^{-3/2}\right] < 0$**, therefore we have a maximum.**

**Solution 2 using Lagrange Multiplier**

**We can write our Lagrange function as** $L(M, L, \lambda) = \frac{2}{3}\left(\sqrt{M} + \sqrt{2L}\right) - \lambda(M + l - 12)$**. Our first-order conditions are**

$$\frac{\partial L}{\partial M} = \frac{1}{3}M^{-\frac{1}{2}} - \lambda = 0$$

$$\frac{\partial L}{\partial L} = \frac{2}{3}(2L)^{-\frac{1}{2}} - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = M + L - 12 = 0$$

**Solving equations 1 and 2 for λ and squaring the Lagrange multiplier implies**

$$\frac{1}{9M} = \frac{4}{9}\left(\frac{1}{2L}\right) \; or \; M = \frac{1}{2}L \; using \; 3rd \; equation, we \; get \; \frac{1}{2}L + L = 12 \; or \; L^* = 8 \; and \; M^* = 4.$$

**At the stationary point, GPA=4. For this problem our determinant of the Bordered Hessian is**

$$|\overline{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -\frac{1}{6}M^{-3/2} & 0 \\ 1 & 0 & -\frac{2}{3}(2L)^{-3/2} \end{vmatrix} = \frac{1}{6}M^{-3/2} + \frac{2}{3}(2L)^{-3/2} > 0$$

**, therefore we have a max.**